Classical equations of motion and scattering amplitudes

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12.05.2022

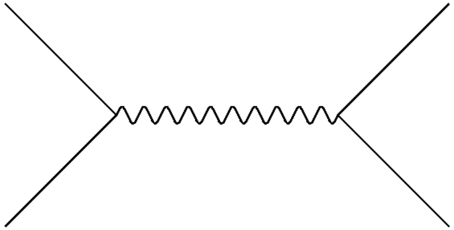
Outline

- Tree level scattering;
- Perturbiner method in flat space;
- Scalars, Gluons, Gravitons;
- Curved backgrounds: (Anti) de Sitter;
- Loop amplitudes;

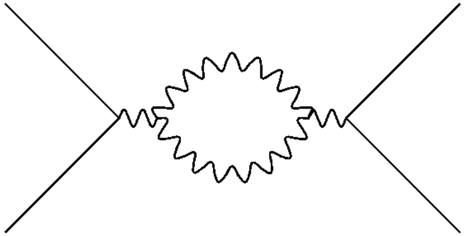
Goal

If you know how to derive an equation of motion,
then you know how to compute tree level scattering!

Scattering amplitudes



- Main observables in QFT.
- Perturbation theory:
  - Tree level (classical).
  - Loops (quantum).



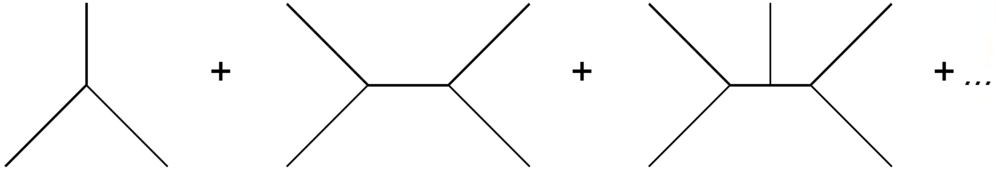
- Different techniques: Feynman diagrams, BCFW, CHY, etc.
- Here I will mostly focus on trees.

Tree graphs and classical fields

Boulware & Brown
1968

Textbook result:

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \Longrightarrow$$



Is there a way to make this easier?

Tree diagrams are recursive by nature*.

*as long as one leg is off-shell.

NEXT SLIDES MIGHT BE A BIT BORING BUT IMPORTANT!

Classical multiparticle solutions

As a warm up: $\mathcal{L} = \frac{1}{2}\Phi\Box\Phi + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3$

Equation of motion: $(\Box + m^2)\Phi = -\frac{\lambda}{2}\Phi^2$

Free case (λ=0): $\Phi(x) = \phi\, e^{ik\cdot x}$ and $k^2 = m^2$

Multiple free particles: $\Phi(x) = \sum_p \phi_p\, e^{ik_p\cdot x}$

It turns out we can solve the full e.o.m. recursively.

$$(\Box + m^2)\Phi = -\tfrac{\lambda}{2}\Phi^2$$

- For example, take: $\Phi(x) = \phi_1 \, e^{ik_1\cdot x} + \phi_2 \, e^{ik_2\cdot x} + \phi_{12} \, e^{ik_{12}\cdot x}$

$$\begin{aligned}(\Box + m^2)\Phi &= (m^2 - k_1^2)\phi_1 \, e^{ik_1\cdot x} + (m^2 - k_2^2)\phi_2 \, e^{ik_2\cdot x} \\ &\quad +(m^2 - k_{12}^2)\phi_{12} \, e^{ik_{12}\cdot x} \\ \Phi^2 &= 2\phi_1\phi_2 e^{ik_{12}\cdot x} + \phi_1^2 e^{2ik_1\cdot x} + \phi_2^2 e^{2ik_2\cdot x}\end{aligned}$$

- We have a solution if: $\phi_{12} = \frac{\lambda}{(k_{12}^2 - m^2)}\phi_1\phi_2 \qquad \phi_1^2 = \phi_2^2 = 0$

This solution generalizes to any number of single-particle states:

$$\Phi(x) = \sum_P \phi_P \, e^{ik_P \cdot x}$$

$$\begin{aligned} k_P^\mu &= k_{p_1}^\mu + \ldots + k_{p_n}^\mu \\ s_P &\equiv k_P^2 \end{aligned}$$

$$\phi_P = \frac{1}{2}\frac{\lambda}{(s_P - m^2)} \sum_{P = Q \cup R} \phi_Q \phi_R$$

- P denotes a ordered “word” formed by single particle labels.

  Ex: $P = 3, \quad P = 25, \quad P = 123, \quad P = 379, \quad P = 1467.$

- The operation P=Q∪R is called a *deshuffle*.

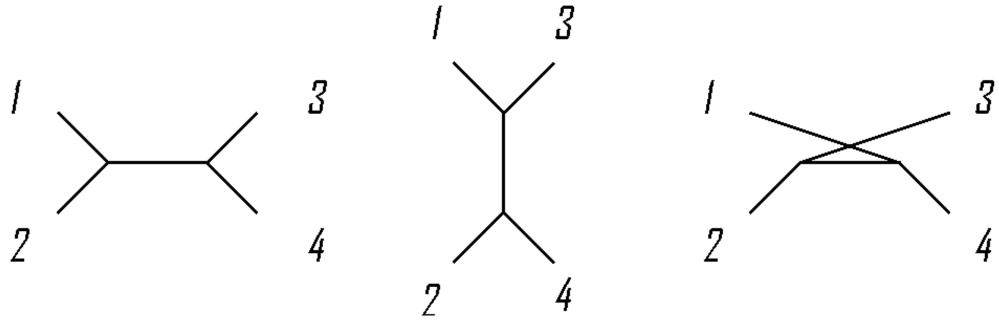
  Ex: $P = 25 \quad \rightarrow \quad (Q, R) = \{(2, 5), (5, 2)\}$

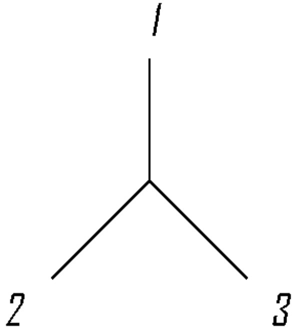
  $P = 123 \quad \rightarrow \quad (Q, R) = \{(1, 23), (2, 13), (3, 12), (23, 1), (12, 3), (13, 2)\}$

So what?! Where are the promised tree level amplitudes??

They are already there, waiting to be harvested.

$$\mathcal{A}_3 \equiv \phi_1(s_{23} - m^2)\phi_{23} = \lambda\phi_1\phi_2\phi_3$$





$$\begin{aligned}\mathcal{A}_4 &\equiv \phi_1(s_{234} - m^2)\phi_{234}\\ &= \lambda\phi_1(\phi_2\phi_{34} + \phi_3\phi_{24} + \phi_4\phi_{23})\\ &= \lambda^2\phi_1\phi_2\phi_3\phi_4\left(\frac{1}{(s_{12} - m^2)} + \frac{1}{(s_{13} - m^2)} + \frac{1}{(s_{14} - m^2)}\right)\end{aligned}$$

Therefore, the multiparticle "currents" recursively defined via e.o.m.

$$\Phi(x) = \sum_P \phi_P \, e^{ik_P \cdot x}$$

$$\phi_P = \frac{1}{2} \frac{\lambda}{(s_P - m^2)} \sum_{P=Q\cup R} \phi_Q \phi_R$$

can be used to compute n-point tree level amplitudes:

$$\mathcal{A}_n = \lim_{s_{2\ldots n} \to m^2} \phi_1 (s_{2\ldots n} - m^2) \phi_{2\ldots n}$$

OBS: No Feynman rules, tracking down factors, combinatorics, signs, ...

STARTING TO GET INTERESTING!

Classical multiparticle solutions: gluons

Equations of motion:
$$\begin{aligned}\partial^\nu F_{\mu\nu} &= i[A^\nu, F_{\mu\nu}] \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\end{aligned}$$

Compared to the scalars discussed, there are two main differences:

1) Quartic interactions:

$$\begin{aligned}\Box A_\mu &= 2i[A^\nu, \partial_\nu A_\mu] - i[A^\nu, \partial_\mu A_\nu] + \underline{[[A_\mu, A_\nu], A^\nu]} \\ &\quad + \partial_\mu(\partial^\nu A_\nu) - i[A_\mu, (\partial^\nu A_\nu)]\end{aligned}$$

2) And?

The gauge symmetry can be used to set the Lorenz gauge

$$\partial^\mu A_\mu = 0$$

such that the e.o.m. we have to solve is simply

$$\Box A_\mu = 2i[A^\nu, \partial_\nu A_\mu] - i[A^\nu, \partial_\mu A_\nu] + [[A_\mu, A_\nu], A^\nu]$$

- Free case:

$$\begin{aligned} \Box A_\mu &= 0 \\ A_\mu(x) &= \epsilon_\mu e^{ik\cdot x} \end{aligned}$$

with

$$k_\mu k^\mu = k_\mu \epsilon^\mu = 0$$
$$\delta\epsilon_\mu = k_\mu \lambda$$

- Multiple free gluons:

$$A^\mu(x) = \sum_p \epsilon_p^\mu \, e^{ik_p\cdot x}$$

And just like in the scalar case, we can have multiparticle solutions:

$$A^\mu(x) = \sum_P \mathcal{A}^\mu_P e^{ik_P\cdot x}$$

$$\mathcal{A}^\mu_P = \frac{1}{s_P}\sum_{P=Q\cup R}[\mathcal{A}^\nu_Q, (2k_{R\nu}\mathcal{A}^\mu_R - k^\mu_R \mathcal{A}_{R\nu})] + \frac{1}{s_P}\sum_{P=Q\cup R\cup S}[\mathcal{A}^\nu_Q, [\mathcal{A}^\mu_R, \mathcal{A}_{S\nu}]]$$

$$k^\mu_P = k^\mu_{p_1} + \ldots + k^\mu_{p_n}$$
$$s_P \equiv k^2_P$$

Berends-Giele currents 1987

- The operation P=Q∪R∪S is also a deshuffle, a simple extension of P=Q∪R.
- This is closer to the original perturbiner formulation.

Rosly & Selivanov 1996-1997

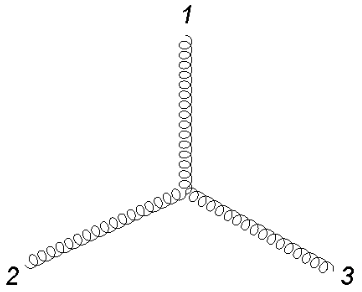
For P=12, we have:

$$
\begin{aligned}
\mathcal{A}^{\mu}_{12} &= \frac{1}{s_{12}} \sum_{12=Q\cup R} [\mathcal{A}^{\nu}_{Q}, (2k_{R\nu}\mathcal{A}^{\mu}_{R} - k^{\mu}_{R}\mathcal{A}_{R\nu})] \\
&+ \frac{1}{s_{12}} \sum_{12=Q\cup R\cup S} [\mathcal{A}^{\nu}_{Q}, [\mathcal{A}^{\mu}_{R}, \mathcal{A}_{S\nu}]]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{\mu}_{12} &= \frac{1}{s_{12}} \left([\epsilon^{\nu}_{1}, (2k_{2\nu}\epsilon^{\mu}_{2} - k^{\mu}_{2}\epsilon_{2\nu})] + [\epsilon^{\nu}_{2}, (2k_{1\nu}\epsilon^{\mu}_{1} - k^{\mu}_{1}\epsilon_{1\nu})] \right) \\
&+ \frac{1}{s_{12}} \sum_{12=Q\cup R\cup S} [\mathcal{A}^{\nu}_{Q}, [\mathcal{A}^{\mu}_{R}, \mathcal{A}_{S\nu}]]
\end{aligned}
$$

The three point amplitude is simply the three gluon vertex:

$$
\begin{aligned}
A(1,2,3) &= \lim_{s_{12}\to 0} \mathrm{Tr}[\epsilon_{3\mu}(s_{12}\mathcal{A}^{\mu}_{12})] \\
&\propto f_{abc}\left\{[(k_2 - k_3)\cdot\epsilon^{a}_{1}](\epsilon^{b}_{2}\cdot\epsilon^{c}_{3}) + \mathrm{cyc}(1,2,3)\right\}
\end{aligned}
$$

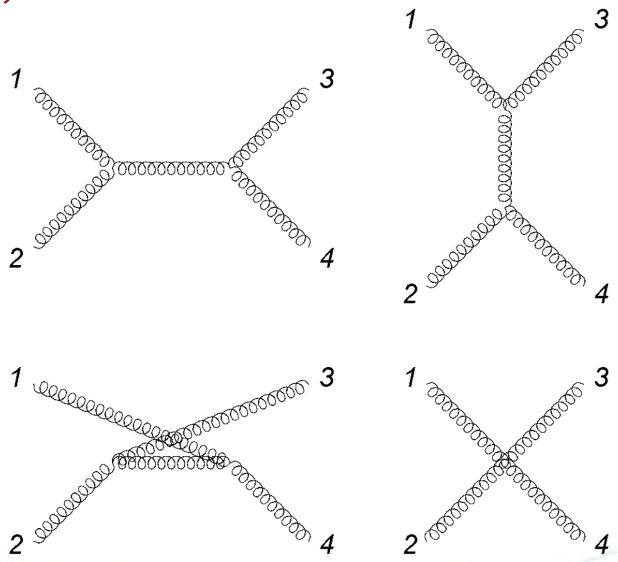


For P=123, we have:

$$\mathcal{A}^{\mu}_{123} = \frac{1}{s_{123}}[\mathcal{A}^{\nu}_{12},(2k_{3\nu}\epsilon^{\mu}_{3}-k^{\mu}_{3}\epsilon_{3\nu})+[\epsilon^{\nu}_{3},(2k_{12\nu}\mathcal{A}^{\mu}_{12}-k^{\mu}_{12}\mathcal{A}_{12\nu})]$$
$$+\frac{1}{s_{123}}[\epsilon^{\nu}_{1},[\epsilon^{\mu}_{2},\epsilon_{3\nu}]]+\text{permutations}(1,2,3)$$

The first line leads to exchange diagrams, while the second is the contact (four-point) one.

$$A(1,2,3,4)=\lim_{s_{123}\to 0}\text{Tr}[\epsilon_{4\mu}(s_{123}\mathcal{A}^{\mu}_{123})]=$$



What I presented so far are called “colour-dressed” perturbiners.

- The are also “colour-stripped” perturbiners: $A^\mu(x) = \sum_P \mathcal{A}^\mu_P e^{ik_P\cdot x} T^{a_P}$
- They lead to partial amplitudes, with a very rich structure.
- Much easier to obtain computationally, and order matters.
- Instead of deshuffles, we have so-called deconcatenations.

Classical multiparticle solutions: gravitons

- Gravity is a whole different matter.

- Einstein field equations: $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

- Gravitons: $$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$$

- Infinity number of vertices! How to even define a recursion?

There is an elegant solution to this problem.

- Suppose there exists a perturbiner for gravity:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_P H_{P\mu\nu} e^{ik_P \cdot x}$$

- Now consider the inverse of the metric, satisfying: $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$

- Then, we can take $g^{\mu\nu}(x) = \eta^{\mu\nu} - \sum_P I_P^{\mu\nu} e^{ik_P \cdot x}$ and find a solution:

$$I_P^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} H_{P\rho\sigma} - \eta^{\nu\sigma} \sum_{P=Q\cup R} I_Q^{\mu\rho} H_{R\rho\sigma}$$

- This way we can avoid the infinity number of vertices in gravity.

A taste of the solution (from ∞ to 5):

$$
\begin{aligned}
\tfrac{s_P}{2} H_{P\mu\nu} \quad = \quad & \sum_{P=Q\cup R} I_Q^{\rho\sigma}(ik_{P\rho}\Gamma_{R\mu\nu\sigma} - ik_{P\nu}\Gamma_{R\mu\rho\sigma}) + \eta^{\alpha\beta}\eta^{\rho\sigma}(\Gamma_{Q\nu\alpha\sigma}\Gamma_{R\mu\rho\beta} - \Gamma_{Q\rho\alpha\sigma}\Gamma_{R\mu\nu\beta}) \\
& + \sum_{P=Q\cup R\cup S} \eta^{\alpha\beta} I_Q^{\rho\sigma}(\Gamma_{R\rho\alpha\sigma}\Gamma_{S\mu\nu\beta} + \Gamma_{R\alpha\rho\beta}\Gamma_{S\mu\nu\sigma} - \Gamma_{R\nu\rho\beta}\Gamma_{S\mu\alpha\sigma} - \Gamma_{R\nu\alpha\sigma}\Gamma_{S\mu\rho\beta}) \\
& + \sum_{P=Q\cup R\cup S\cup T} I_Q^{\rho\sigma} I_R^{\alpha\beta}(\Gamma_{S\nu\alpha\sigma}\Gamma_{T\mu\rho\beta} - \Gamma_{S\rho\alpha\sigma}\Gamma_{T\mu\nu\beta})
\end{aligned}
$$

- n-point graviton amplitudes: (unimaginable using diagrams, including matter interactions!)

$$
\begin{aligned}
\mathcal{M}_n \quad & \equiv \quad \kappa \lim_{s_{2\ldots n}\to 0} s_{2\ldots n}\, h_{1\mu\nu} I^{\mu\nu}_{2\ldots n} \\
& = \quad \kappa \lim_{s_{2\ldots n}\to 0} s_{2\ldots n}\, h_1^{\mu\nu} H_{2\ldots n\mu\nu}
\end{aligned}
$$

- Phys.Rev.Lett. 127 (2021) 18, 181603

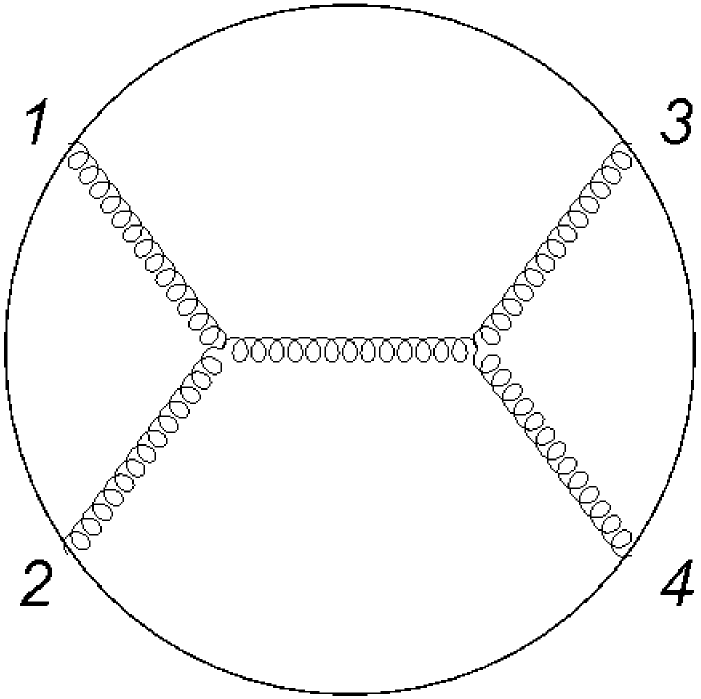
ONGOING PROJECTS

(Anti) de Sitter

- Scalar e.o.m.: $$\begin{aligned} g^{mn}\nabla_m\nabla_n\phi &= m^2\phi + V(\phi) \\ g^{mn}\nabla_m\nabla_n\phi &= g^{mn}\partial_m\partial_n\phi - g^{mn}\Gamma^p_{mn}\partial_p\phi \\ &= \tfrac{1}{R^2}[z^2\partial_z^2 + (1-d)z\partial_z\phi + z^2\eta^{\mu\nu}\partial_\mu\partial_\nu]\phi \end{aligned}$$
- Free solutions: $\phi(z,x) = \mathcal{K}(k,z)e^{ik\cdot x}$
- Multiparticle ansatz: $\phi(z,x) = \sum_P \underline{\Phi_P(z)}e^{ik_P\cdot x}$
- Generates Witten diagrams (scalars, gluons, gravitons)!

$$
\begin{aligned}
(\mathcal{D}_P^2 - m^2)\mathcal{A}_{P\mu} &= izk_{P\mu}[z\partial_z + (2-d)]\alpha_P \\
&+i(z\partial_z - d)\sum_{P=QR}(\alpha_Q \mathcal{A}_{R\mu} - \alpha_R \mathcal{A}_{Q\mu}) \\
&-z\sum_{P=QR}\{(k_{Q\mu}\alpha_Q + i\partial_z \mathcal{A}_{Q\mu})\alpha_R - (k_{R\mu}\alpha_R + i\partial_z \mathcal{A}_{R\mu})\alpha_Q\} \\
&-z\sum_{P=QR}\{(k_{Q\mu} - k_{R\mu})(\mathcal{A}_Q \cdot \mathcal{A}_R) + 2\mathcal{A}_{R\mu}(k_R \cdot \mathcal{A}_Q) - 2\mathcal{A}_{Q\mu}(k_Q \cdot \mathcal{A}_R)\} \\
&+\sum_{P=QRS}\{(\alpha_R \mathcal{A}_{S\mu} - \alpha_S \mathcal{A}_{R\mu})\alpha_Q + \mathcal{A}_{S\mu}(\mathcal{A}_Q \cdot \mathcal{A}_R) - \mathcal{A}_{R\mu}(\mathcal{A}_Q \cdot \mathcal{A}_S)\} \\
&-\sum_{P=QRS}\{(\alpha_Q \mathcal{A}_{R\mu} - \alpha_R \mathcal{A}_{Q\mu})\alpha_S + \mathcal{A}_{R\mu}(\mathcal{A}_Q \cdot \mathcal{A}_S) - \mathcal{A}_{Q\mu}(\mathcal{A}_R \cdot \mathcal{A}_S)\} \\
\alpha_P &= \tfrac{2}{s_P z}\sum_{P=QR}\{\alpha_R(k_R \cdot \mathcal{A}_Q) - \alpha_Q(k_Q \cdot \mathcal{A}_R)\} \\
&+\tfrac{i}{s_P z}\sum_{P=QR}\{(\mathcal{A}_Q \cdot \partial_z \mathcal{A}_R) - (\mathcal{A}_R \cdot \partial_z \mathcal{A}_Q)\} \\
&+\tfrac{1}{s_P z^2}\sum_{P=QRS}\{2\alpha_R(\mathcal{A}_Q \cdot \mathcal{A}_S) - \alpha_S(\mathcal{A}_Q \cdot \mathcal{A}_R) - \alpha_Q(\mathcal{A}_R \cdot \mathcal{A}_S)\}
\end{aligned}
$$

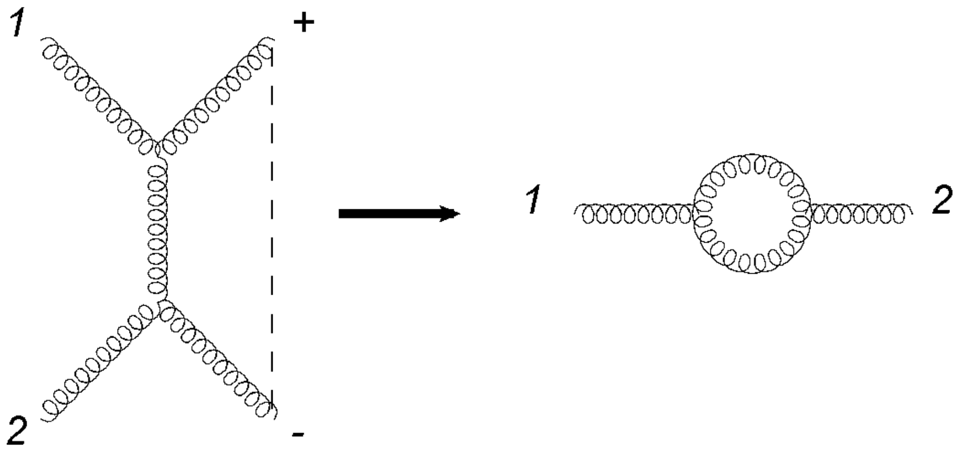
$$
\begin{aligned}
A(1,2,3,4) \;=\; & \int dz \Big\{ \tfrac{1}{z^{d+1}} \left[(\mathcal{A}_1 \cdot \mathcal{A}_4)(\mathcal{A}_2 \cdot \mathcal{A}_3) - (\mathcal{A}_1 \cdot \mathcal{A}_3)(\mathcal{A}_2 \cdot \mathcal{A}_4) \right] \\
& + \partial_z \left[\tfrac{1}{z^{d+1}} \tfrac{1}{s_{34}} [(\mathcal{A}_3 \cdot \partial_z \mathcal{A}_4) - (\mathcal{A}_4 \cdot \partial_z \mathcal{A}_3)](\mathcal{A}_1 \cdot \mathcal{A}_2) \right] \\
& + \tfrac{1}{s_{34}} \tfrac{1}{z^{d+1}} [(\mathcal{A}_1 \cdot \partial_z \mathcal{A}_2) - (\mathcal{A}_2 \cdot \partial_z \mathcal{A}_1)][(\mathcal{A}_3 \cdot \partial_z \mathcal{A}_4) - (\mathcal{A}_4 \cdot \partial_z \mathcal{A}_3)] \\
& - \tfrac{1}{z^{d+1}} \tfrac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{s_{34}} \left[z (\mathcal{A}_1 \cdot \mathcal{A}_2) \right] \frac{1}{(\mathcal{D}_{34}^2 - m^2)} \left[z (\mathcal{A}_3 \cdot \mathcal{A}_4) \right] \\
& - \tfrac{\eta^{\mu\nu}}{z^{d+1}} \left[z \left(2\mathcal{A}_{2\mu} (k_2 \cdot \mathcal{A}_1) - 2\mathcal{A}_{1\mu} (k_1 \cdot \mathcal{A}_2) + (k_{1\mu} - k_{2\mu})(\mathcal{A}_1 \cdot \mathcal{A}_2) \right) \right] \frac{1}{(\mathcal{D}_{34}^2 - m^2)} \times \\
& \times \left[z \left(2\mathcal{A}_{4\nu} (k_4 \cdot \mathcal{A}_3) - 2\mathcal{A}_{3\nu} (k_3 \cdot \mathcal{A}_4) + (k_{3\nu} - k_{4\nu})(\mathcal{A}_3 \cdot \mathcal{A}_4) \right) \right] \Big\} \\
& - [(34) \to (23)].
\end{aligned}
$$

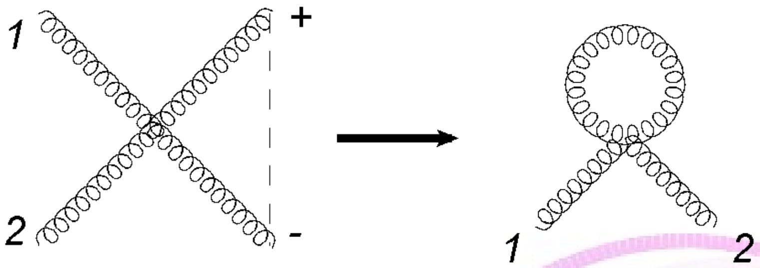


1
3
2
4

Loops are trees too!

The perturbiner can also generate trees with off-shell legs:



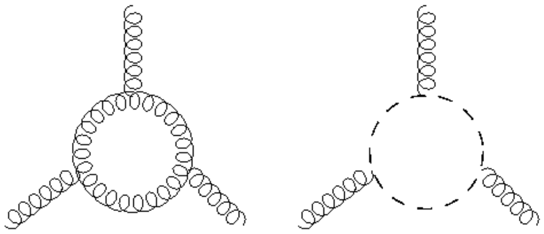


We can do this consistently for gluons and gravitons (including ghosts).

In case you are curious: BV actions and ghosts

- Yang-Mills theory:

$$S = \int d^dx \mathrm{Tr}\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu) - \partial^\mu \bar{c}(\partial_\mu c - i[A_\mu, c])\}$$



- Gravity:

$$S = \frac{1}{2\kappa}\int d^dx[\sqrt{-g}R + (\xi^\rho\partial_\rho g^{\mu\nu} - 2g^{\mu\rho}\partial_\rho\xi^\nu)g^*_{\mu\nu} + (\xi^\rho\partial_\rho\xi^\mu)\xi^*_\mu] + \chi^*_\mu\Lambda^\mu]$$

Final remarks

- Classical e.o.m. are intimately connected to scattering amplitudes;
- Elegant and compact computations (and easy to code);
- Encompasses a broad set of theories (including gravity!);
- Extension to curved spaces (AdS, to appear soon);
- Off-shell recursions and loop integrands (to appear soon);
- Rich underlying structure (L-infinity and A-infinity algebras);
- Interplay with string theory: Phys.Rev.Lett. 127 (2021) 5, 051601;

Thank you!