A Classical-Quantum Correspondence (CQC)

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arXiv: 1803.08919, 1806.05196, 1807.10282

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arXiv: 1904.12962

Motivation

Quantum effects in classical backgrounds —

★ Hawking radiation during gravitational collapse.

★ Schwinger pair creation.

★Coupling of classical inflaton to other quantum fields.

★ etc.

Expand field in modes:

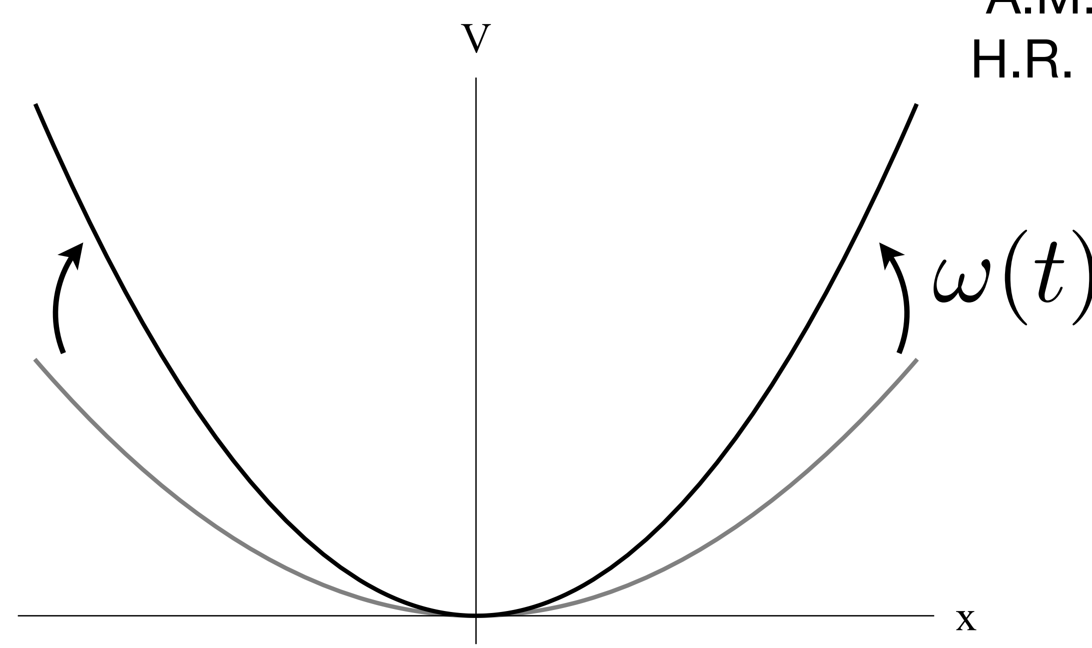
$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \Big(c_{\mathbf{k}}(t) f_{\mathbf{k}}(\mathbf{x}) + c^{\dagger}_{\mathbf{k}}(t) f^*_{\mathbf{k}}(\mathbf{x}) \Big)$$

*For free fields, the mode coefficients are **simple harmonic oscillator** variables in a time dependent classical background.*

Simple Harmonic Oscillator

(with time dependent frequency)

Early Work:
H.R. Lewis, 1968
A.M. Perelmov & V.S. Popov, 1969
H.R. Lewis & W.B. Riesenfeld, 1969
L. Parker, 1971



Define ladder operators:

$$\hat{a} = \frac{\hat{p} - im\omega(t)\hat{x}}{\sqrt{2m\omega(t)}}, \quad \hat{a}^\dagger = \frac{\hat{p} + im\omega(t)\hat{x}}{\sqrt{2m\omega(t)}}$$

SHO contd.

Heisenberg equations: $$\frac{d\hat{a}}{dt} = -i[\hat{a}, H] + \frac{\partial \hat{a}}{\partial t}$$ $$[\hat{a}, \hat{a}^\dagger] = 1$$

Solution: $$\hat{a}(t) = \frac{(p_z^* - im\omega z^*)}{\sqrt{2m\omega}}\hat{a}_0 + \frac{(p_z - im\omega z)}{\sqrt{2m\omega}}\hat{a}_0^\dagger$$

"Bogolyubov coefficients"

where,

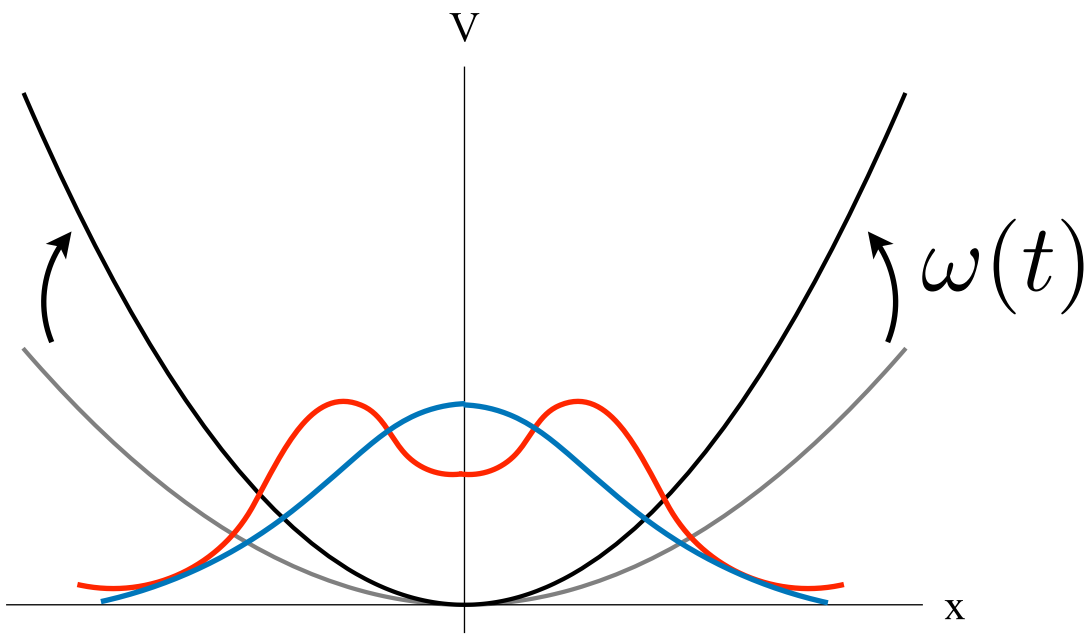
$$\boxed{\ddot{z} + \omega^2(t) z = 0}$$

z is complex!

with initial conditions

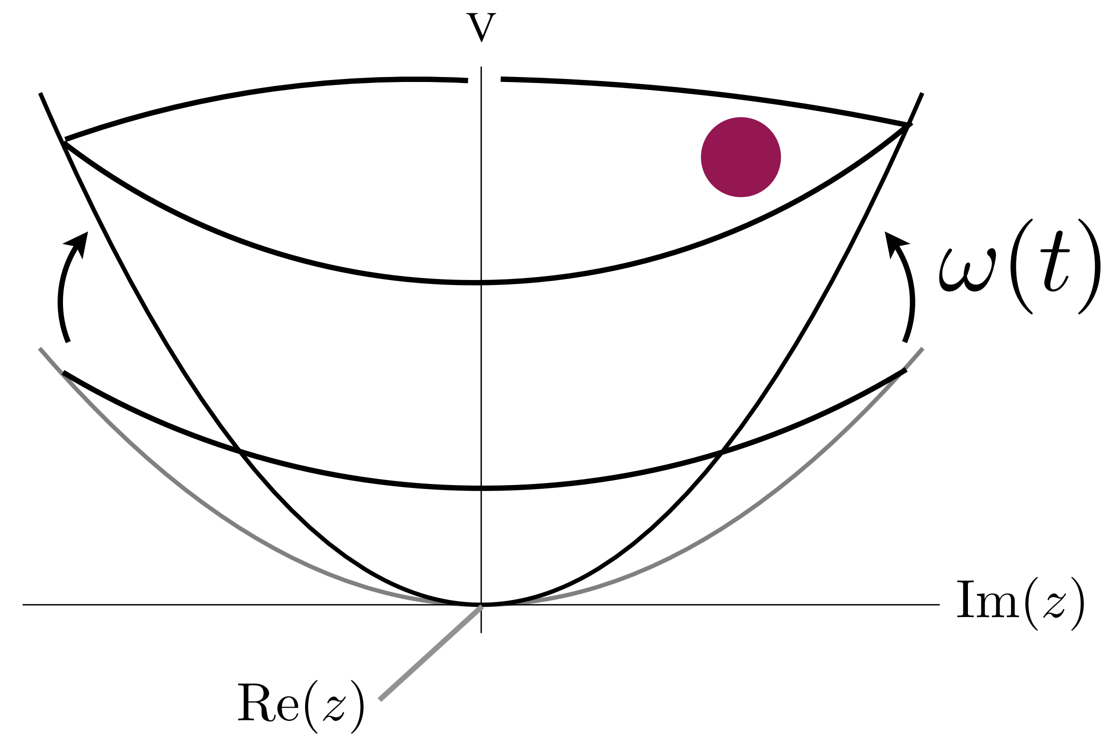
$$\boxed{z(0) = \frac{-i}{\sqrt{2m\omega_0}}, \quad \dot{z}(0) = \sqrt{\frac{\omega_0}{2m}}}$$

Classical-Quantum Correspondence (CQC)



Quantum

$$\longleftrightarrow$$



Classical

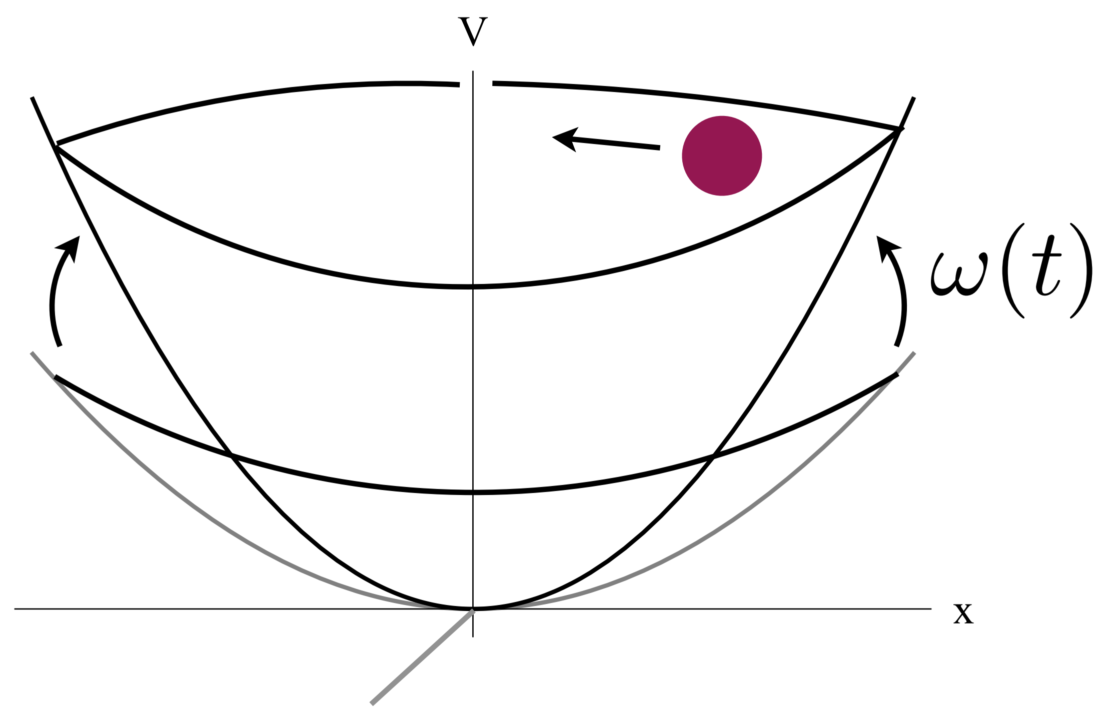
$$\hat{x} \leftrightarrow z$$

$$|0\rangle \leftrightarrow \{z_0, \dot{z}_0\}$$

Note 1: for any ω(t)!

Note 2: all quantum operators can be written in terms of the classical variable z(t) and the initial values of the operators.

Initial Conditions



Quantum ground state implies classical system must have:

- zero point energy = $\omega/2$
- angular momentum = 1/2

(In field theory, angular momentum corresponds to global charge.)

Particle Production with CQC

Particle production is usually discussed via Bogolyubov transformations.

Using CQC, we can find particle production from a classical calculation:

$$E_{\text{radiation}} = \omega \left(\underbrace{|\beta|^2}_{\text{Bogolyubov coefficient}} + \frac{1}{2} \right) = \frac{|p_z|^2}{2m} + \frac{m\omega^2}{2} |z|^2$$

Summary: Particle production in classical time-dependent backgrounds, (*e.g.* Hawking radiation, reheating during inflation,…) can *all* be calculated using a classical analysis.

This result was known from the late 60's.
We now want more…

Popov & Perelmov
Zeldovich & Starobinsky
Berger
B.-L. Hu
Hu & Parker…

Backreaction with CQC

Particle production implies backreaction on the classical background.

- slow-down of inflaton dynamics,
- evaporation of black holes,
- screening of electric fields,
- etc..

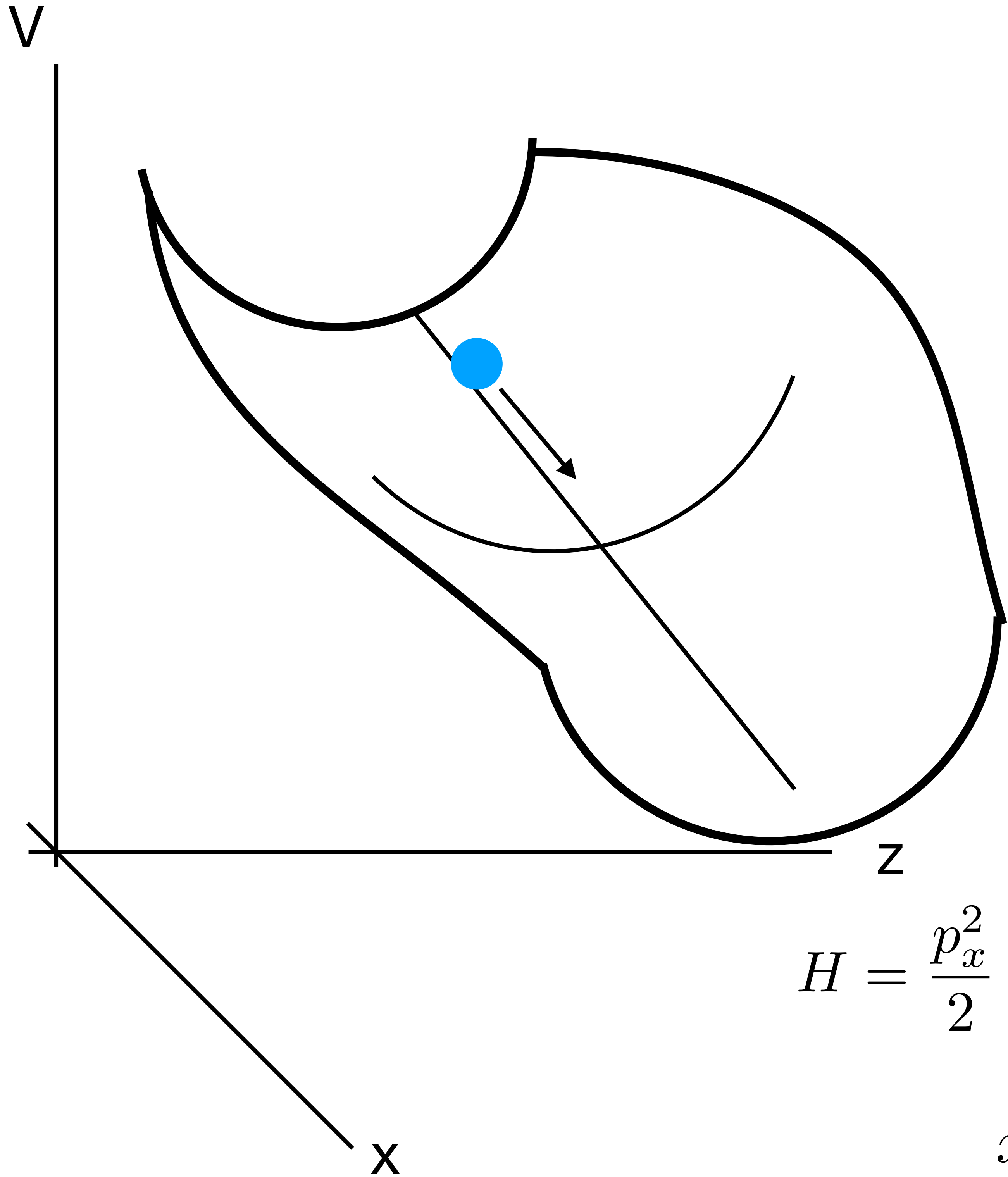
Cooper & Mottola,
Kluger et al,
Y-Z. Chen & TV, …

CQC provides a classical framework to address quantum backreaction.

Three Examples:

- Rolling background.
- Quantum oscillon.
- Collapsing shell.

Example 1: Rolling Background



$$H = \frac{p_x^2}{2} - ax + \frac{p_z^2}{2} + \frac{1}{2}\omega_0^2 z^2 + \frac{\lambda}{2}x^2 z^2$$

$$x_{cl} = \frac{1}{2}at^2, \quad z_{cl} = 0$$

Analysis

CQC Analysis: $\ddot{x} = a - \lambda x|z|^2, \qquad \ddot{z} = -(\omega_0^2 + \lambda x^2)z$ (z=complex)

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad z_R(0) = 0, \quad \dot{z}_R(0) = \sqrt{\frac{\omega_0}{2}}, \qquad z_I(0) = \frac{-1}{\sqrt{2\omega_0}}, \quad \dot{z}_I(0) = 0$$

Simple numerical problem — takes few seconds with Mathematica.

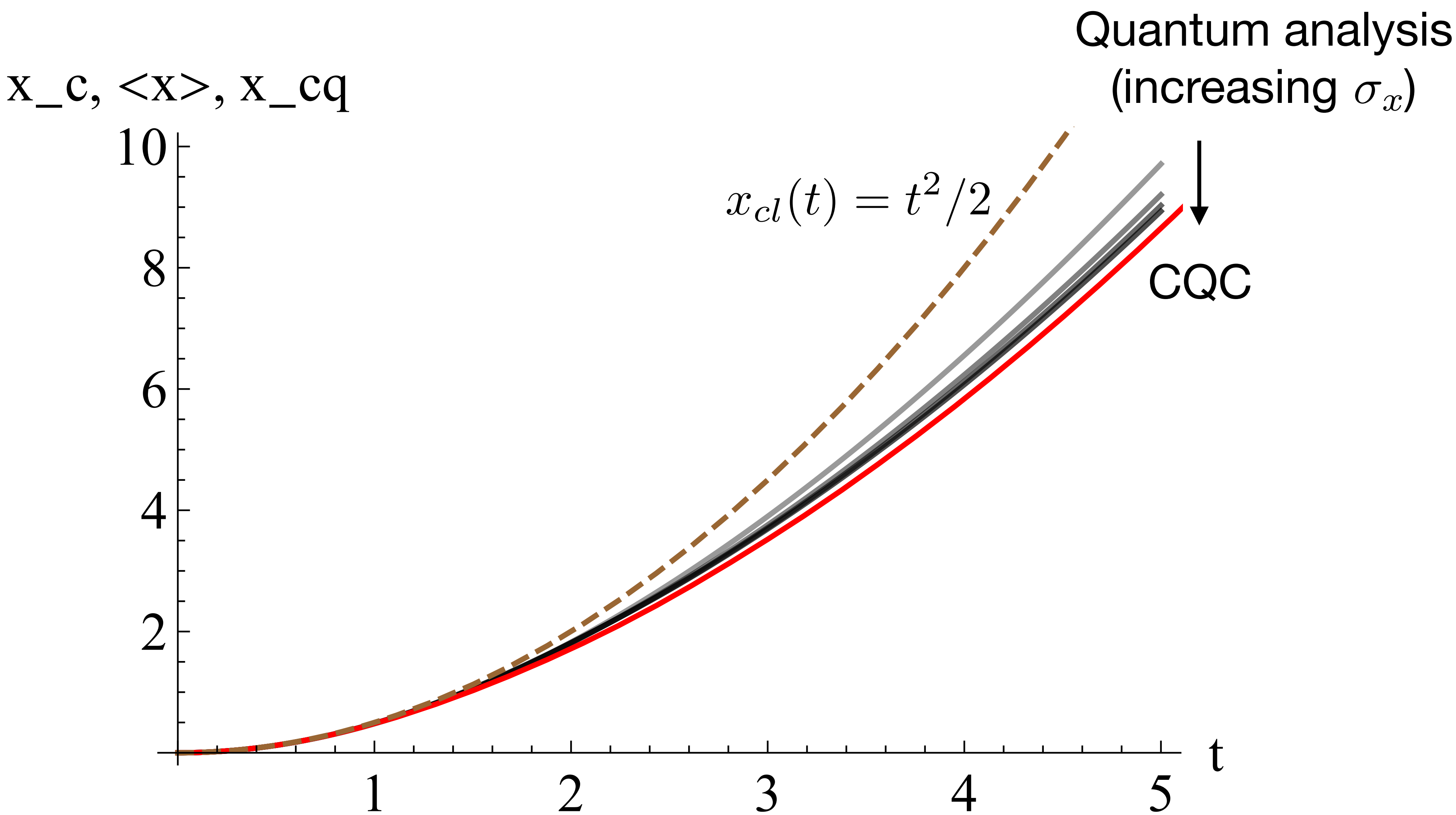
Full Quantum Analysis: $H\psi(x,z,t) = i\frac{\partial}{\partial t}\psi(x,z,t)$ (z=real)

$$\psi(t=0,x,z) = \left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} e^{-x^2/(2\sigma_x^2)} \times \left(\frac{\omega_0}{\pi}\right)^{1/4} e^{-\omega_0 z^2/2}$$

free parameter

Challenging and laborious numerical problem — takes several days on cluster.

Particle on linear potential



Approach to CQC

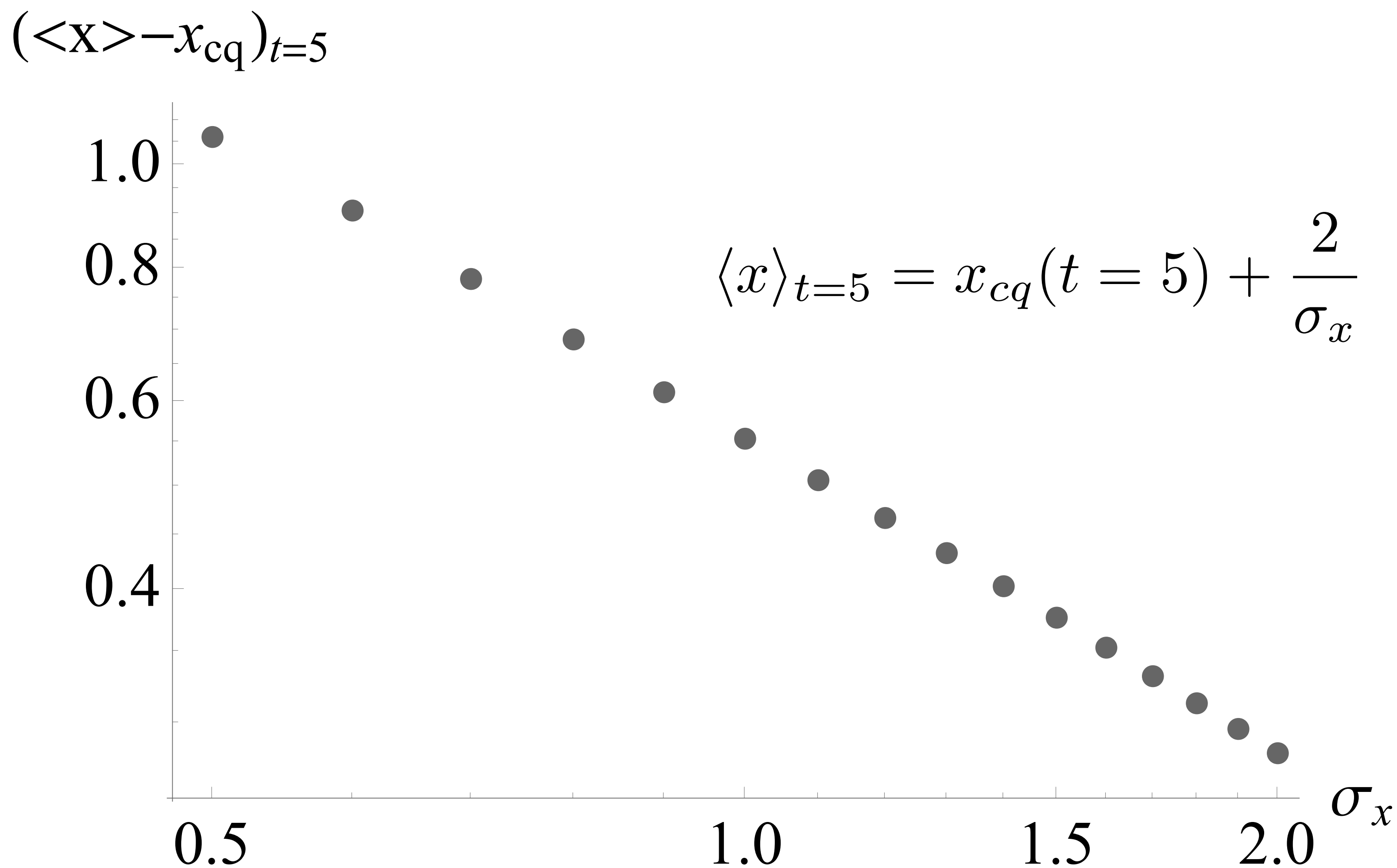
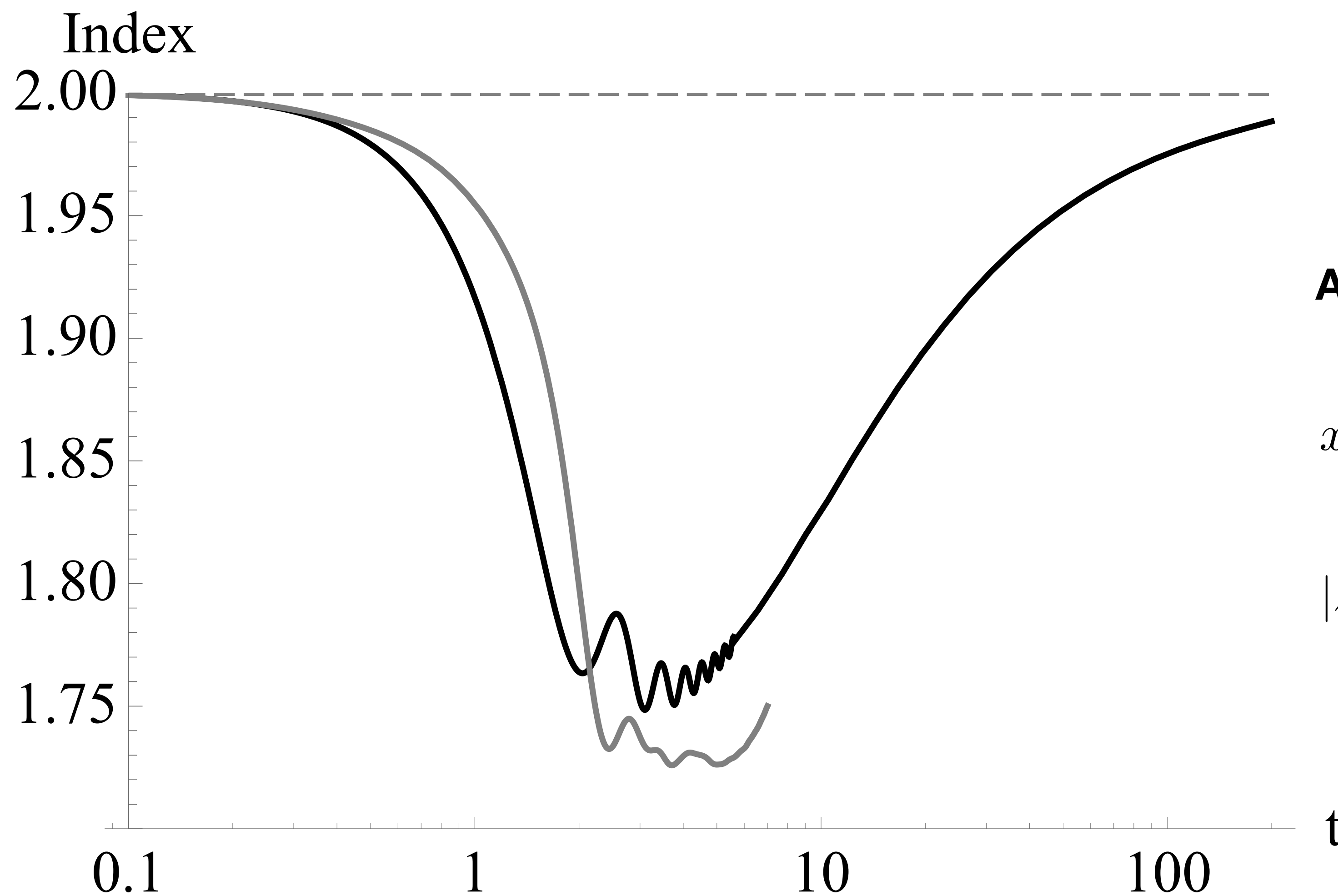


FIG. 4: Log-log plot of $\langle x\rangle - x_{cq}$ at $t = 5$ showing that the CQC becomes more exact for larger σ_x.

Particle on linear potential: exponent



Asymptotic CQC:

$$x(t) = \frac{1}{2}\left(a - \frac{\sqrt{\lambda}}{2}\right)t^2$$

$$|z(t)|^2 = \frac{1}{\sqrt{\lambda}\left(a - \frac{\sqrt{\lambda}}{2}\right)t^2}$$

FIG. 5: Log-linear plot of the scaling index $n_s = d\ln(f)/d\ln(t)$ for $f = x_{cq}$ (in black) and $f = \langle x\rangle_{\sigma_x=2}$ (in gray). At late times the scaling index of x_{cq} approaches 2.

Quantum vs. CQC

CQC is *exact* if the background is classical.

The wavepacket of the rolling particle moves down the potential and spreads.

The dynamics is classical if the rolling is faster than the spreading:

$$at \gg \frac{\hbar}{2m\sigma_x}$$

Therefore quantum dynamics coincides with CQC for wide wavepackets at late times.

CQC for fields

Coupled fields: $$S = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\Phi^2\phi^2\right]$$

Quantum fields in curved space-time: $$S = \int d^4x \sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

General discretized version:

$$S_{\text{discrete}} = \sum_{K,L}\left[\frac{1}{2}\dot{\phi}_K\mathcal{M}_{KL}\dot{\phi}_L - \frac{1}{2}\phi_K\mathcal{N}_{KL}\phi_L\right]$$

We will work with discretization on a spatial lattice: x=n*a,
as this has certain advantages over a mode decomposition.

Same as N coupled quantum simple harmonic oscillators.

CQC for fields contd.

Details may be found in “Classical Quantum Correspondence for Fields”w **George Zahariade**.
(arXiv:1807.10282)

$$\{\phi_K\} \to Z_{IJ} \qquad \text{complex NxN matrix}$$

$i.e.$ N field variables correspond to $2N^2$ classical variables.

$$S_c = \int dt \frac{1}{2a} \mathrm{Tr}\left[\dot{\mathrm{Z}}^\dagger \dot{\mathrm{Z}} - \mathrm{Z}^\dagger \Omega^2 \mathrm{Z}\right]$$

$$a^2\mathbf{\Omega}^2 = \begin{pmatrix} 2+\lambda a^2\Phi_1^2 & -1 & 0 & 0 & 0 & \ldots \\ -1 & 2+\lambda a^2\Phi_2^2 & -1 & 0 & 0 & \ldots \\ 0 & -1 & 2+\lambda a^2\Phi_3^2 & -1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Initial conditions: $Z_0 = -i\sqrt{\frac{a}{2}}\sqrt{\Omega_0}^{-1}, \ \ \dot{Z}_0 = \sqrt{\frac{a}{2}}\sqrt{\Omega_0}$

CQC for fields contd.

$$\ddot{Z}_{ij} + \Omega_{ik}^2 Z_{kj} = 0$$

$$\ddot{\Phi}_i - \frac{1}{a^2}\left(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}\right) + V'(\Phi_i) + \frac{\lambda}{a^2}\sum_{j=1}^{N} Z_{ij}^* Z_{ij}\Phi_i = 0$$

Initial conditions: $Z_0 = -i\sqrt{\frac{a}{2}}\sqrt{\Omega_0}^{-1}, \quad \dot{Z}_0 = \sqrt{\frac{a}{2}}\sqrt{\Omega_0}$

Proof of CQC

TV & Zahariade (fields paper)

Semiclassical equation for background:

$$\Box\Phi + V(\Phi) + \lambda\langle 0|\phi^2|0\rangle\Phi = 0$$

Expectation value known in terms of Z (lattice form):

$$\langle 0|\phi^2|0\rangle\Big|_{x=ia} = \frac{1}{a^2}\sum_{j=1}^{N} Z^*_{ij}[\Phi]Z_{ij}[\Phi]$$

which gives the CQC equation for the background:

$$(\Box\Phi)_i + V'(\Phi_i) + \frac{\lambda}{a^2}\sum_{j=1}^{N} Z^*_{ij}Z_{ij}\Phi_i = 0$$

Alternately, can show that the iterative semiclassical approximation gives the CQC in the limit of infinite iterations.

Example 2: Evaporation of Quantum Oscillon

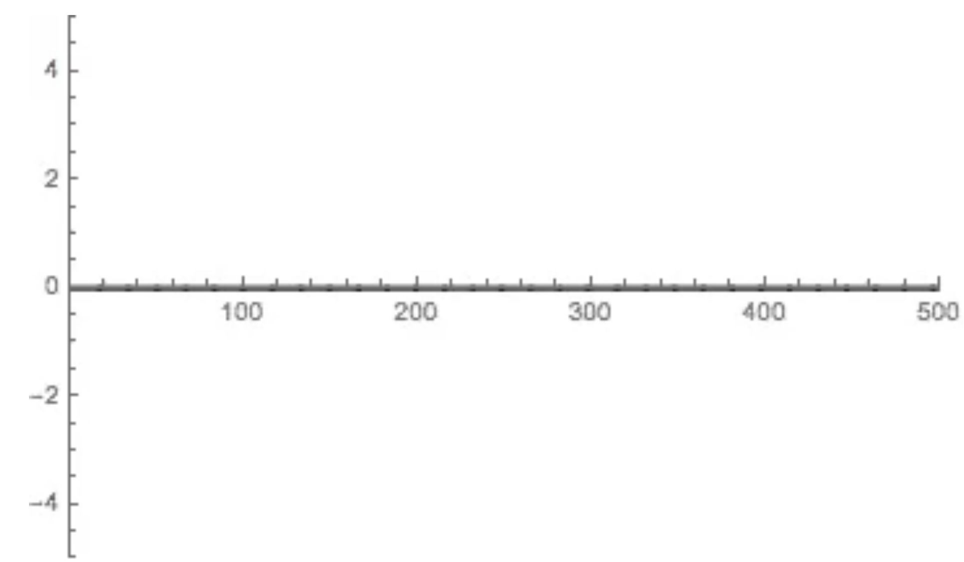
also Hertzberg (2010)

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - m_\phi^2(1-\cos\phi) + \frac{1}{2}(\partial_\mu \psi)^2 - \frac{\lambda}{2}\phi^2\psi^2$$

Breather solution (exact):

$$\phi_b(t,x) = 4\,\tan^{-1}\left[\frac{\eta\sin(\omega t)}{\cosh(\eta\omega x)}\right], \quad \psi = 0$$

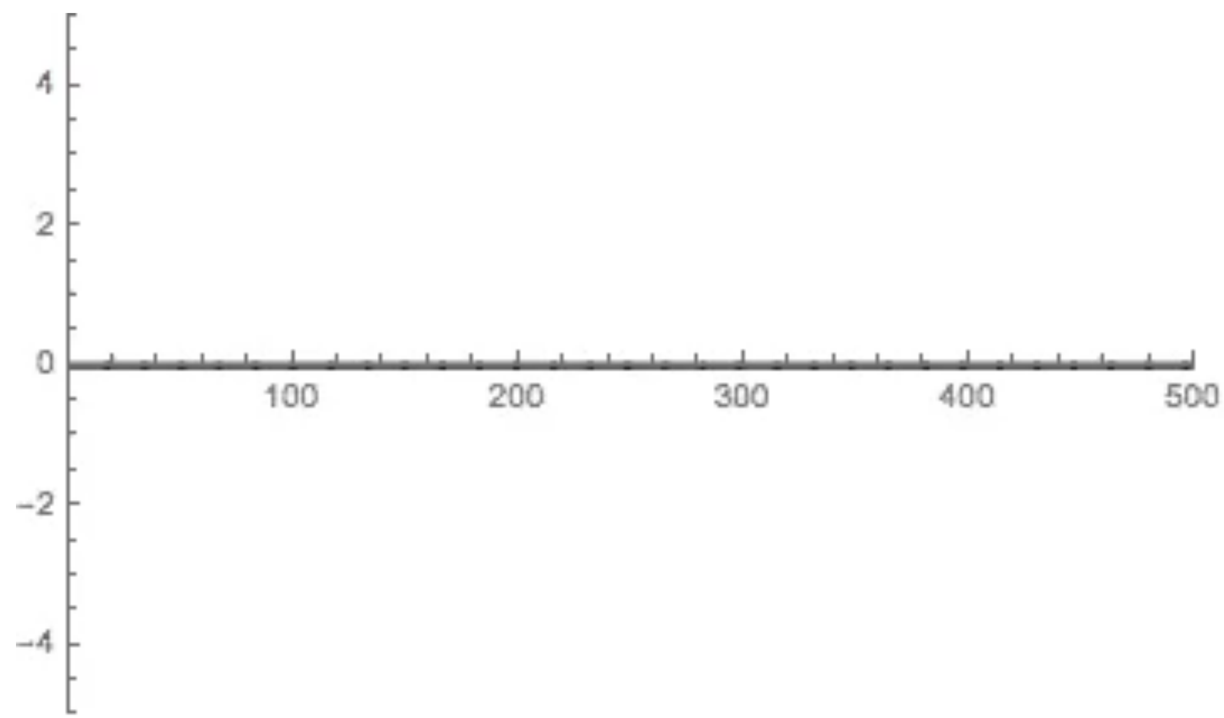
$$\omega\eta = \sqrt{1-\omega^2}$$



Breather will evaporate due to quantum radiation, *i.e.* “quantum oscillon”.

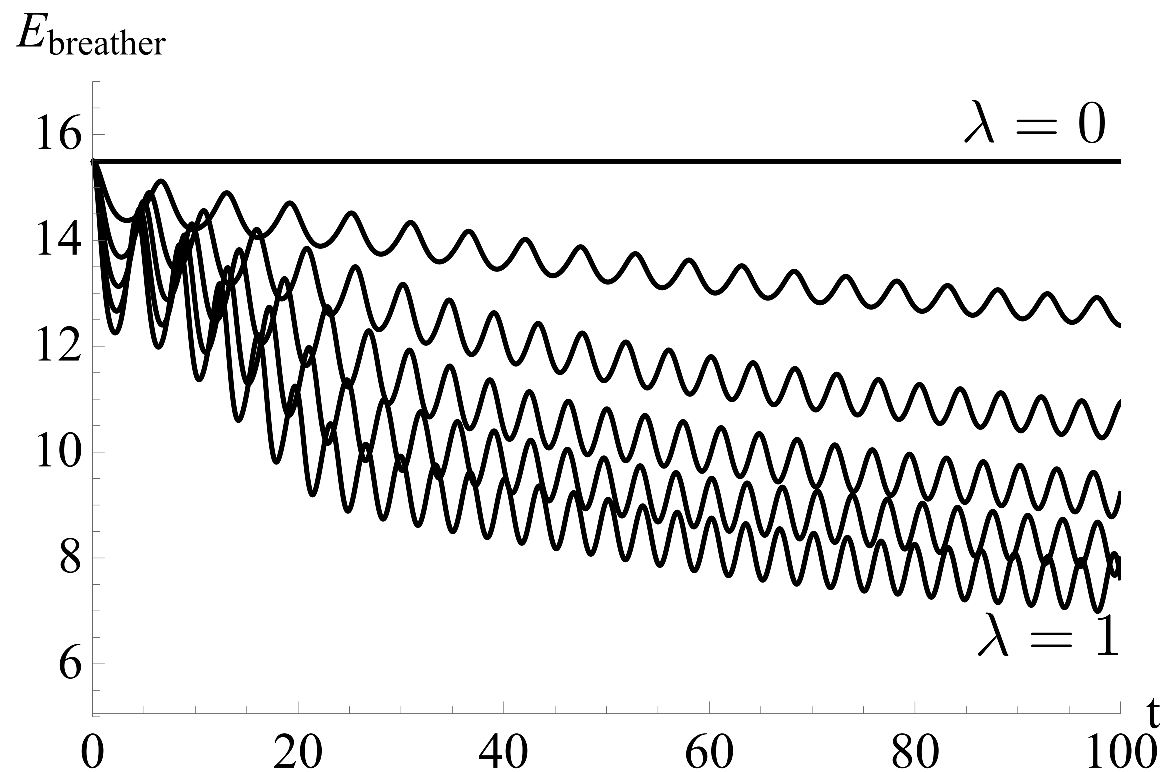
Apply CQC to evaporation.
(No other technique is available to my knowledge.)

Quantum oscillon: evaporation



Quantum Oscillon via CQC

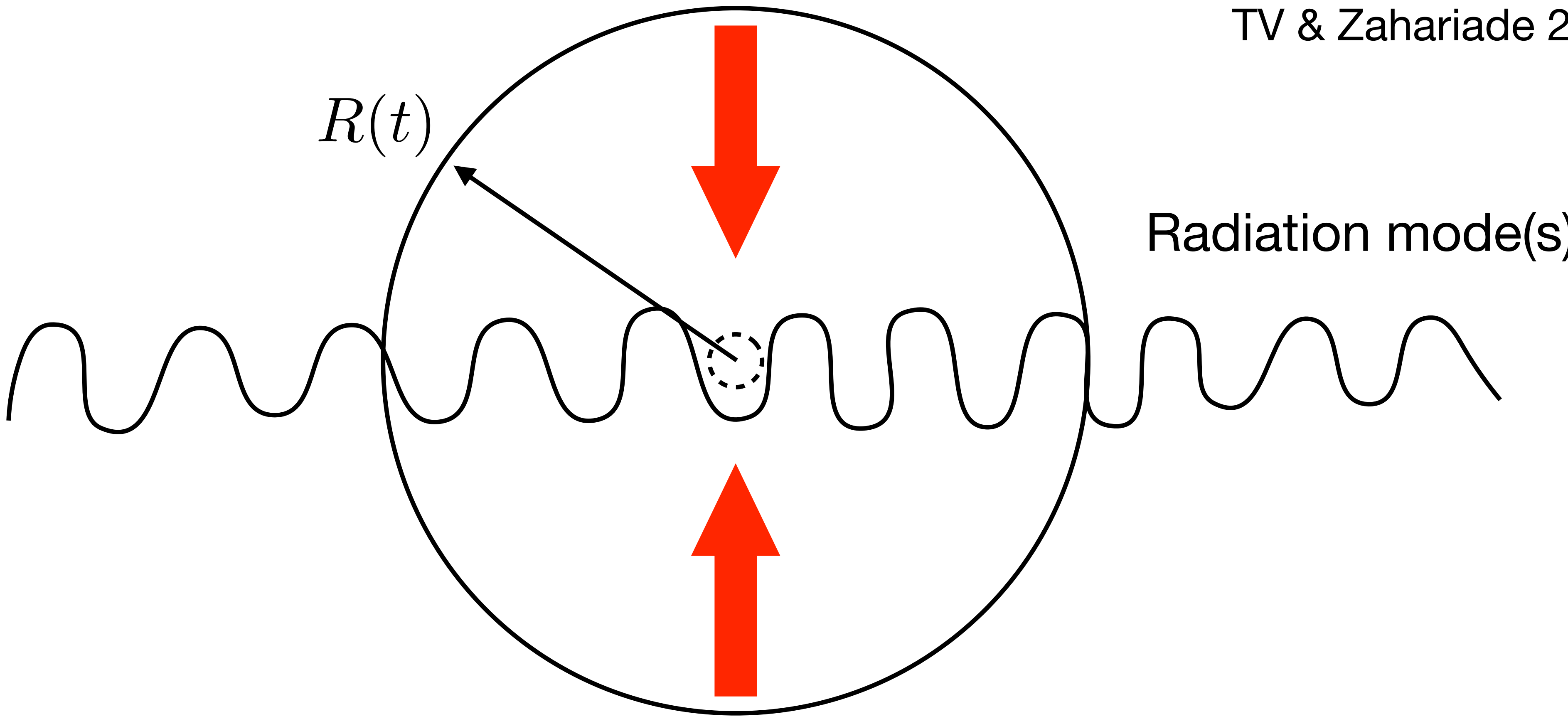
with **Jan Olle, Oriol Pujolas, George Zahariade**



Renormalization is necessary to compare evolutions for different lattice spacings.

Example 3: Collapsing Shell

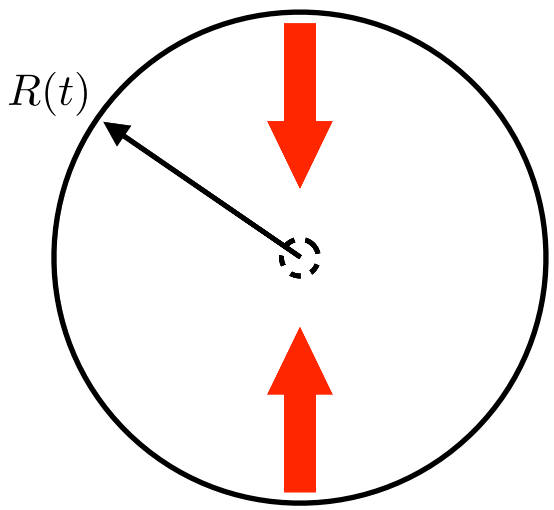
TV, Stojkovic & Krauss, 2007
Kolopanis & TV, 2013
TV & Zahariade 2018



Changing shell metric leads to quantum radiation and shell evaporation.

Recast as simple harmonic oscillators with time-dependent frequencies.

Shell Dynamics



Ipser & Sikivie, 1984

$$g_{\mu\nu} = \begin{cases} \text{Minkowski } (T, r, \theta, \phi), & r \leq R(T) \\ \text{Schwarzschild } (t, r, \theta, \phi), & r > R(T) \end{cases}$$

*Scalar field changes the form of the metric but for now we restrict attention to just the shell radius and the field excitations.

$$\dot{T} \equiv \frac{dT}{dt} = \frac{B}{\sqrt{B + (1-B)R_T^2}} \qquad B \equiv 1 - \frac{2GM}{R} = 1 - \frac{R_S}{R}$$

$$\boxed{\dot{T} \to 0 \text{ as } R \to R_S.}$$

Scalar field in shell metric

TV, Stojkovic & Krauss, 2007

Inside shell: $$S_{\rm in} = -2\pi \int dT \int_0^{R(T)} r^2 dr \left(-(\partial_T \phi)^2 + (\partial_r \phi)^2 \right)$$

Outside shell:

$$S_{\rm out} = -2\pi \int dT \int_{R(T)}^{\infty} r^2 dr \left(-\frac{\dot{T}}{1 - 2GM/r} (\partial_T \phi)^2 + \frac{1 - 2GM/r}{\dot{T}} (\partial_r \phi)^2 \right)$$

Take near-horizon limit. Only keep dominant terms.

$$S_\phi \approx 2\pi \int dT \left(\int_0^{2GM} r^2 dr (\partial_T \phi)^2 - \int_{2GM}^{\infty} r^2 dr \left(1 - 2GM/r\right) \frac{(\partial_r \phi)^2}{\dot{T}} \right)$$

Scalar field modes in shell metric

Expand scalar field in modes. Since action is quadratic, the modes will be given by simple harmonic oscillator variables.

Diagonalize using principal-axis transformation:

$$S_\phi \approx \sum_k \int dT \left(\frac{1}{2} {q'_k}^2 - \frac{{\lambda_k}^2}{2\dot{T}} q_k^2 \right) \qquad ' \equiv \frac{d}{dT}$$

For now, treat only one mode.

$$S_\phi \to \int dT \left(\frac{1}{2} {q'}^2 - \frac{\kappa^2}{2\dot{T}} q^2 \right)$$

Toy model for shell+1 excitation

Now with the CQC!

$$S = -4\pi\sigma \int dT R^2 \left[\underbrace{\sqrt{1 - R_T^2} - 2\pi G\sigma R}_{\text{Shell dynamics.}} \right] + \frac{1}{2\omega_0} \int dT \left[\underbrace{\frac{|z'|^2}{2} - \frac{\kappa^2 |z|^2}{2\dot{T}}}_{\text{One radiation mode.}} \right]$$

$$\boxed{\omega^2(T) = \frac{\kappa^2}{\dot{T}} \to \infty, \quad \text{as } R \to R_S.}$$

Next we solve the classical equations with CQC initial conditions.

Evaporation of Gravitating Collapsing Shell

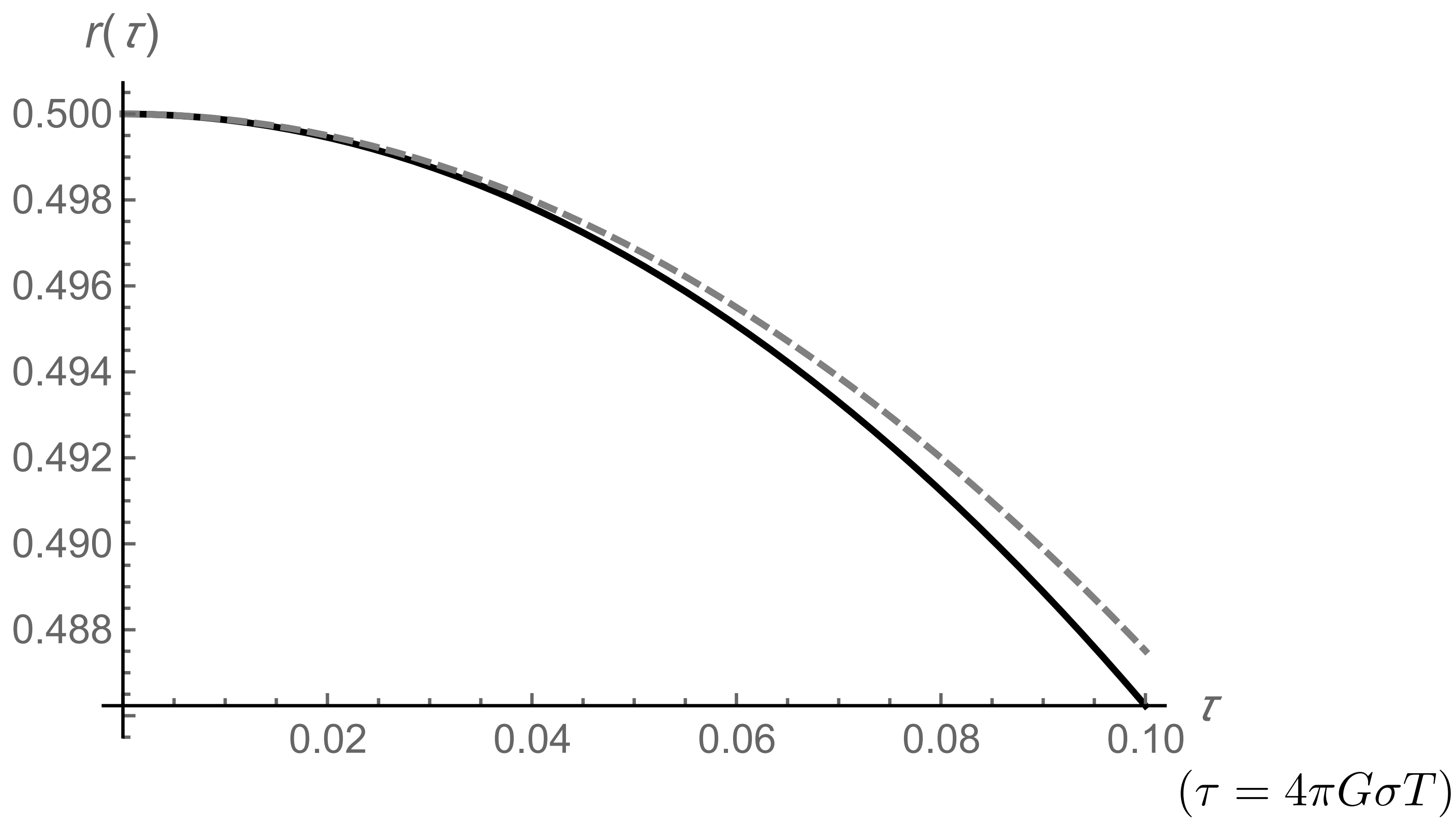
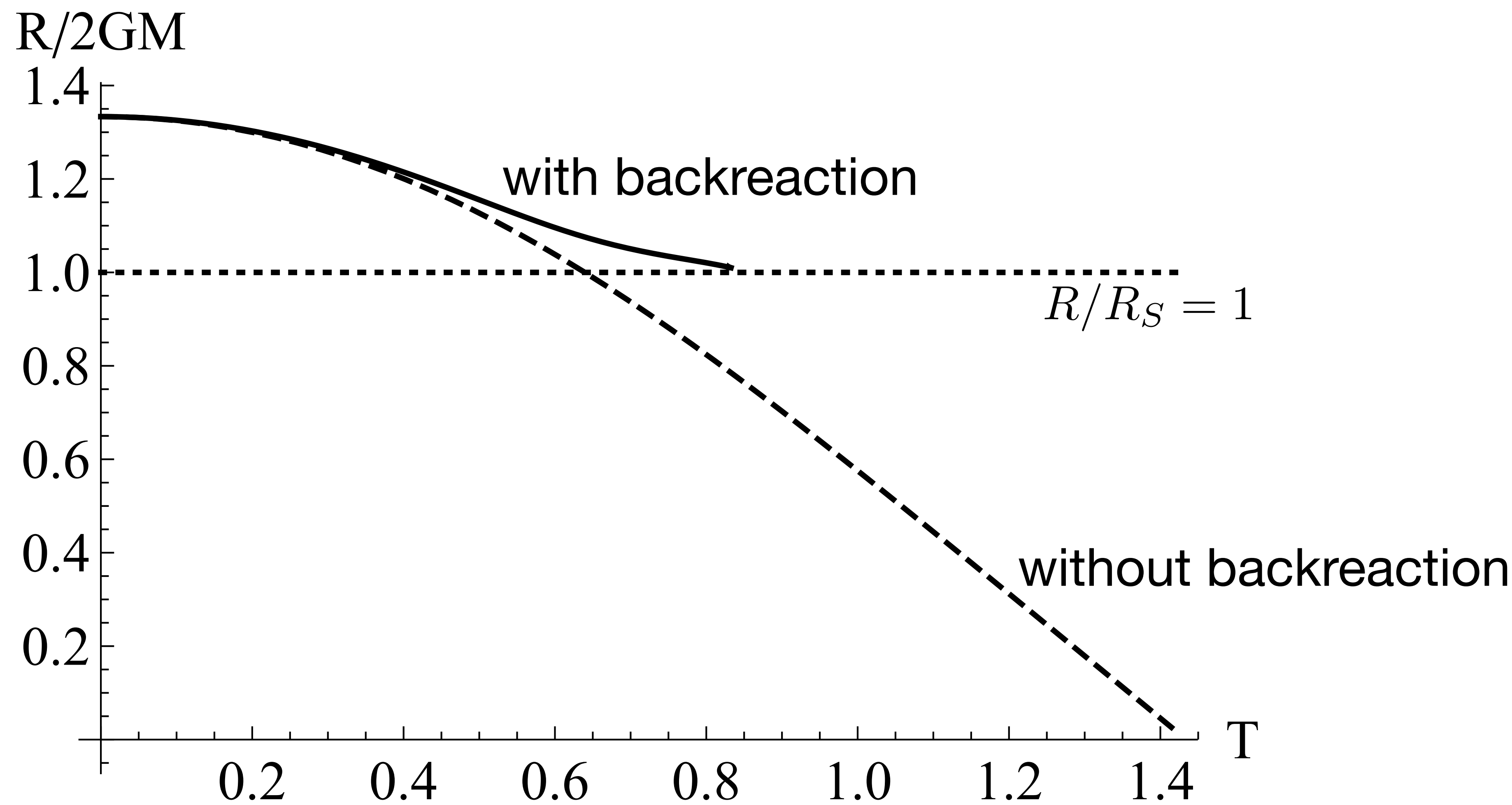


Figure 1. Radius of shell r versus time τ when backreaction is ignored (dashed curve) and with backreaction taken into account (solid curve).

Shell Evaporation with CQC



Conclusions

★ **Classical Quantum Correspondence (CQC):** New (!) technique to study quantum systems in classical backgrounds.

★ **With CQC we can address backreaction:** Comparison with full quantum backreaction in the rolling problem shows excellent agreement.

★ **Quantum oscillons:** Quantum evaporation using the CQC**.**

★ **Gravitational collapse:** Single-mode-CQC hints that collapse slows down with respect to instantaneous Schwarzschild radius. A full implementation seems within reach.