Chaos in the collective dynamics of **nuclei**

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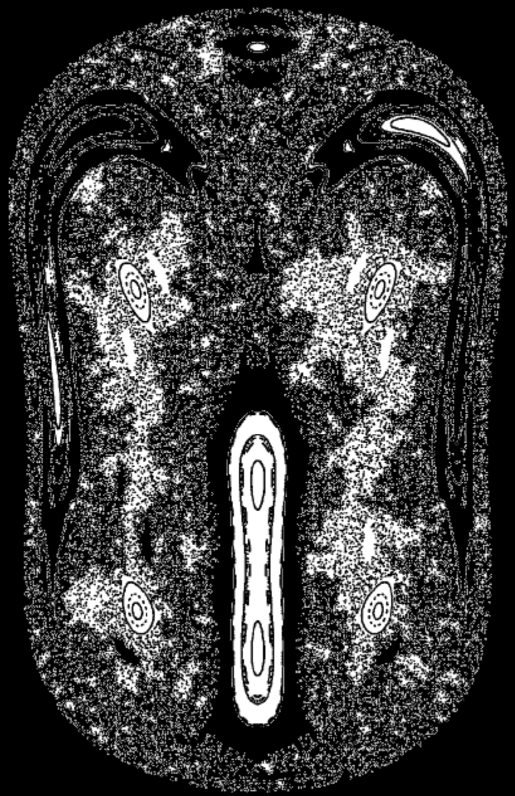
With essential contribution of:
Pavel Stránský (Mexico → Prague)
Michal Macek (Jerusalem → Yale)

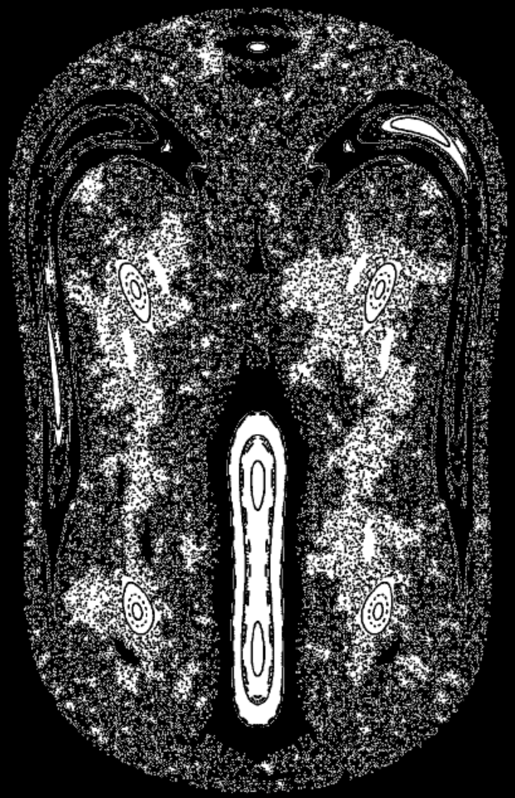


FZU AV CR, Praha
April 2014

Understanding chaos *via nuclei*

Outline of the Fuga:

1) Preludium
   compound nucleus & random matrices
2) Exposition
   classical chaos & collective models of nuclei
3) Development
   quantum chaos & collective models of nuclei
4) Postludium
   coding & decoding physics



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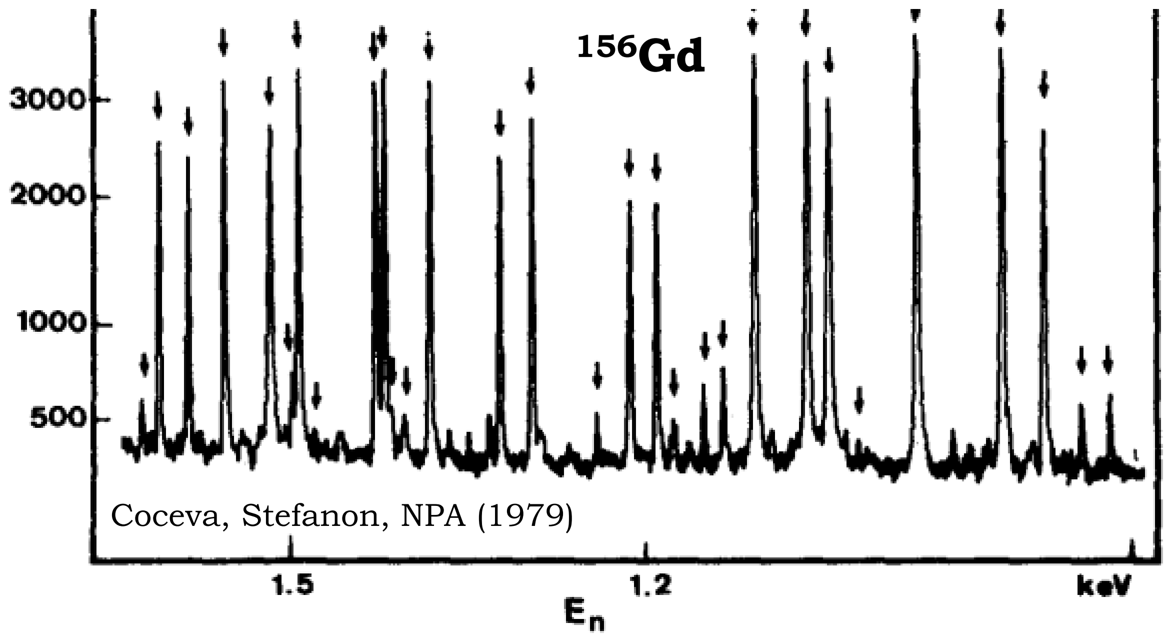
Compound nucleus

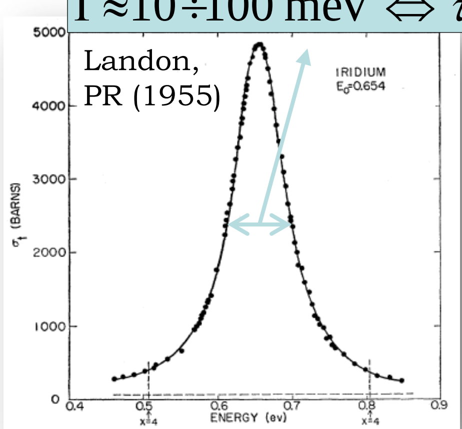
Niels Bohr 1936

Resonances in the neutron-nucleus cross sections:

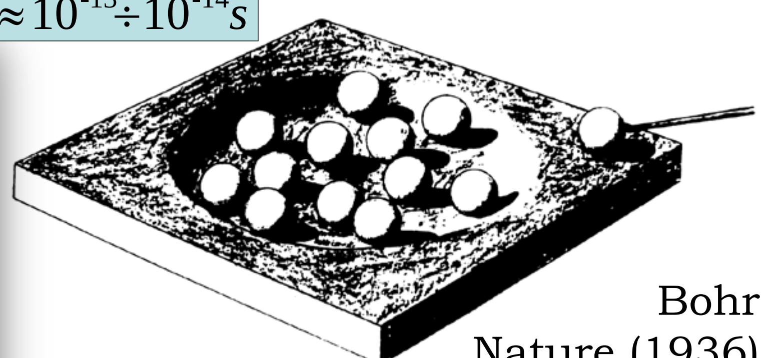


Coceva, Stefanon, NPA (1979)

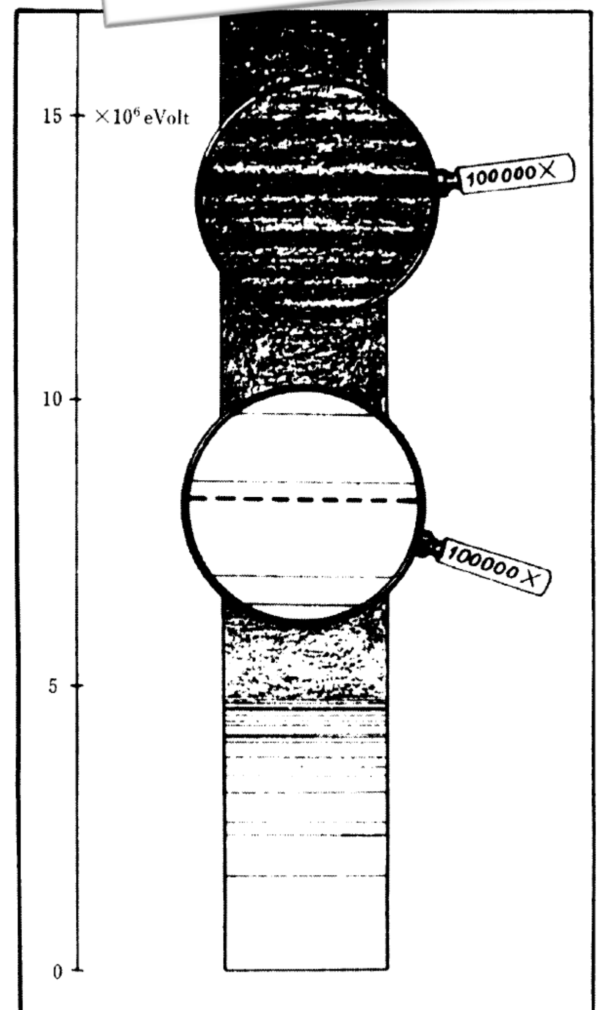
$\Gamma \approx 10 \div 100\ \mathrm{meV} \Leftrightarrow \tau \approx 10^{-13} \div 10^{-14} s$



Landon, PR (1955)



Bohr Nature (1936)



Random Hamiltonians **Wigner 1955**

How to describe neutron resonances?
No chance of detailed theory! Only a **statistical** approach possible.

Take the Hamiltonian at random!

$$\hat{H} \equiv \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix}$$

Hermitian

$H_{ij} = H_{ji} \in \mathbb{R}$ real ...invariance
$H_{ij} = H_{ji}^{*} \in \mathbb{C}$ complex ...noninvariance
} under **time-reversal**

Task: Find distribution $W(\hat{H})$ in the space of matrices such that the probability $W(\hat{H})d\hat{H}$ is invariant under unitary/orthonormal transformations $\hat{U}$ or $\hat{O}$ (for simplicity assume statistical independence of matrix elements)

Solution: **Gaussian Unitary/Orthogonal Ensemble = GUE / GOE**

$$W(\hat{H}) = N \exp\left[-b \operatorname{Tr} \hat{H}^2\right] = N \exp\left[-b\left(\sum_i H_{ii}^2 + \sum_{i>j} (\operatorname{Re} H_{ij})^2 + \sum_{i>j} (\operatorname{Im} H_{ij})^2\right)\right]$$

Note 1: This distribution maximizes the entropy $S = -\int W(\hat{H}) \ln W(\hat{H}) d\hat{H}$ under constraint $\int \operatorname{Tr} \hat{H}^2 W(\hat{H})\, d\hat{H} = \text{const}$

Note 2: We use metric and measure in the space of matrices:

$$ds^2 = \operatorname{Tr}[\delta\hat{H}\delta\hat{H}^{+}] = \sum_i \delta H_{ii}^2 + 2\sum_{i>j} |\delta H_{ij}|^2$$

invariant under unitary/orthonormal transformations

$$d\hat{H} = \underbrace{\sqrt{|\det g_{\mu\nu}|}}_{\substack{2^{n(n-1)/4}\ \text{real} \\ 2^{n(n-1)/2}\ \text{complex}}} \prod_i dH_{ii} \prod_{i>j} d \operatorname{Re} H_{ij} \underbrace{\left\{\prod_{i>j} d \operatorname{Im} H_{ij}\right\}}_{\text{complex}}$$

Random Hamiltonians **Wigner 1955**

We look for the statistical distribution of the set of eigenvalues

Diagonalization transformation depending on some "angles"

$$\hat{H} \equiv \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \xrightarrow{\hat{U} \text{ or } \hat{O}} \begin{pmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_n \end{pmatrix}$$

- repulsive **Coulomb potential** in **2D** (force ~ $1/r$)
- **oscillator trap**

$$P(E_1, E_2 \cdots E_n) = N \overbrace{\prod_{i>j} \left|E_i - E_j\right|^{\beta}}^{\text{Jacobian}} \exp\left[-b \sum_i E_i^2\right] \propto \exp\left[-b \sum_i E_i^2 - \beta \sum_{i>j} \ln\left|E_i - E_j\right|\right]$$

Thermal distribution of equally **charged 2D particles** in an oscillator trap at inverse temperature:

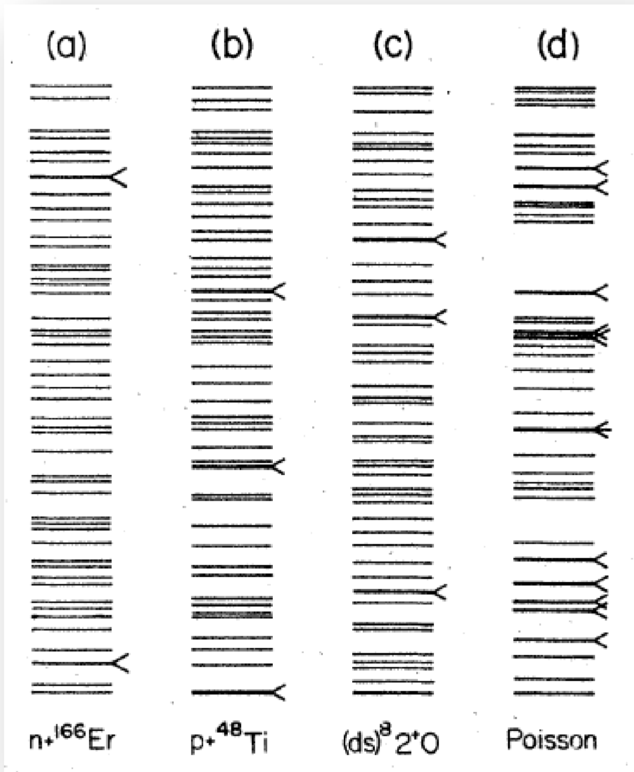
$$\beta = \begin{cases} 1 & \text{GOE} \\ 2 & \text{GUE} \end{cases}$$

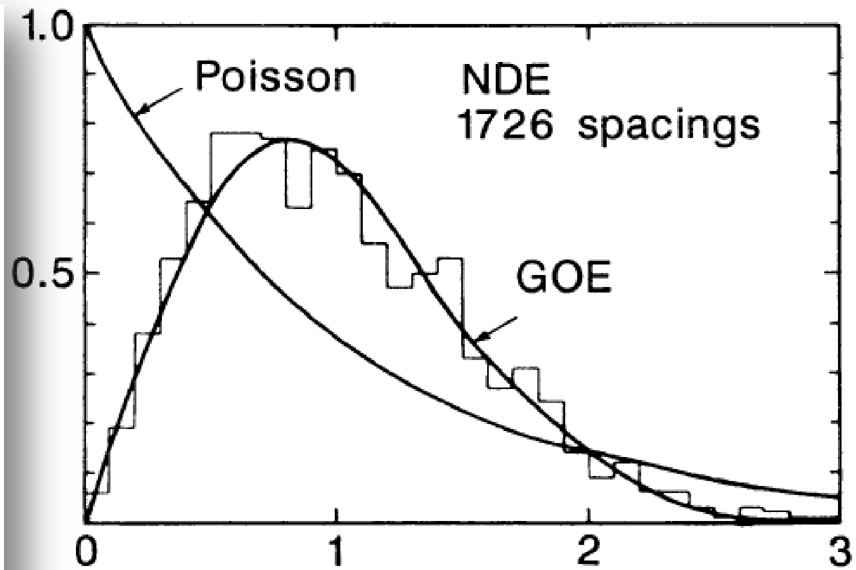
⇒ **repulsion of neighboring levels** (for levels with the same conserved…

⇒ **correlations mediated by Coulomb interactions** …quantum numbers)

Contrast: particles with no charges – Poisson distribution with no correlations

Spectral correlations in nuclei





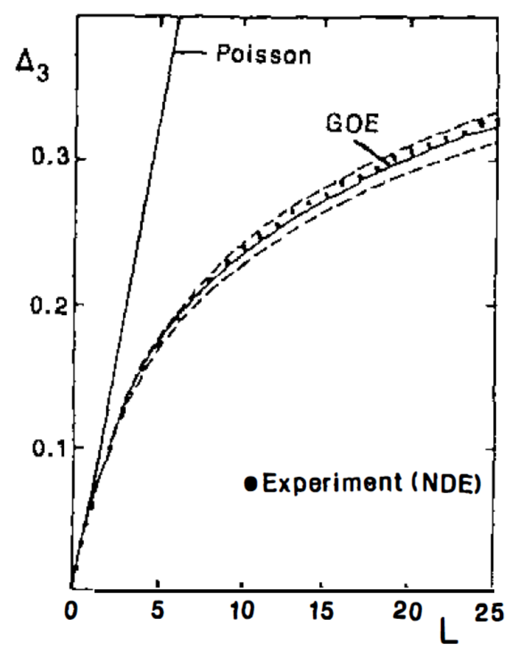
- Haq, Pandey, Bohigas, PRL 48,1086 (1982)
- Bohigas, Haq, Pandey, in Nuclear Data for Science and Technology (Dordrecht, Reidel, 1983), p. 809

Nuclear Data Ensemble

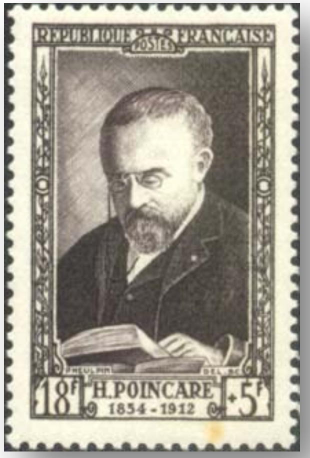
Neutron Resonances
complete sequences of typically □150-170 resonances with the same spin & parity in one heavy nucleus

Proton Resonances
shorter (□60-80) complete sequences of the same spin & parity in one heavy nucleus

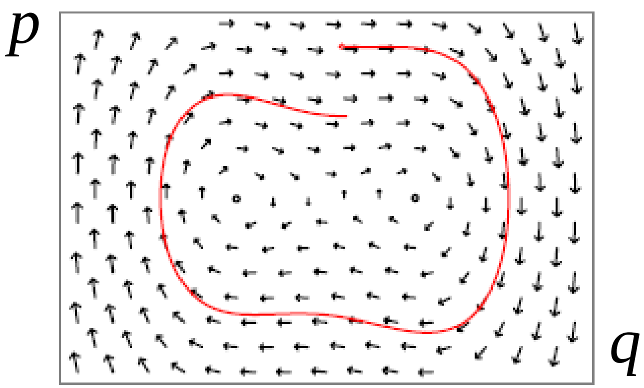
=> In total, 1726 spacings



Non-integrable systems

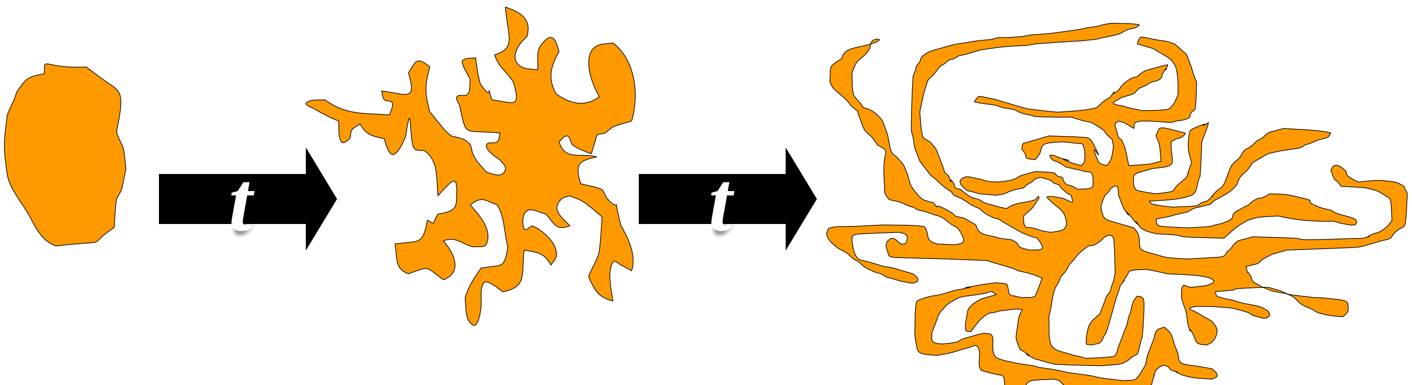


$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$



Trajectories
$\rightarrow$ flow in the phase space

"Incompressible fluid":
volume is conserved (Lioville theorem $\frac{d}{dt}\rho = 0$)
but **shape** can become very complicated

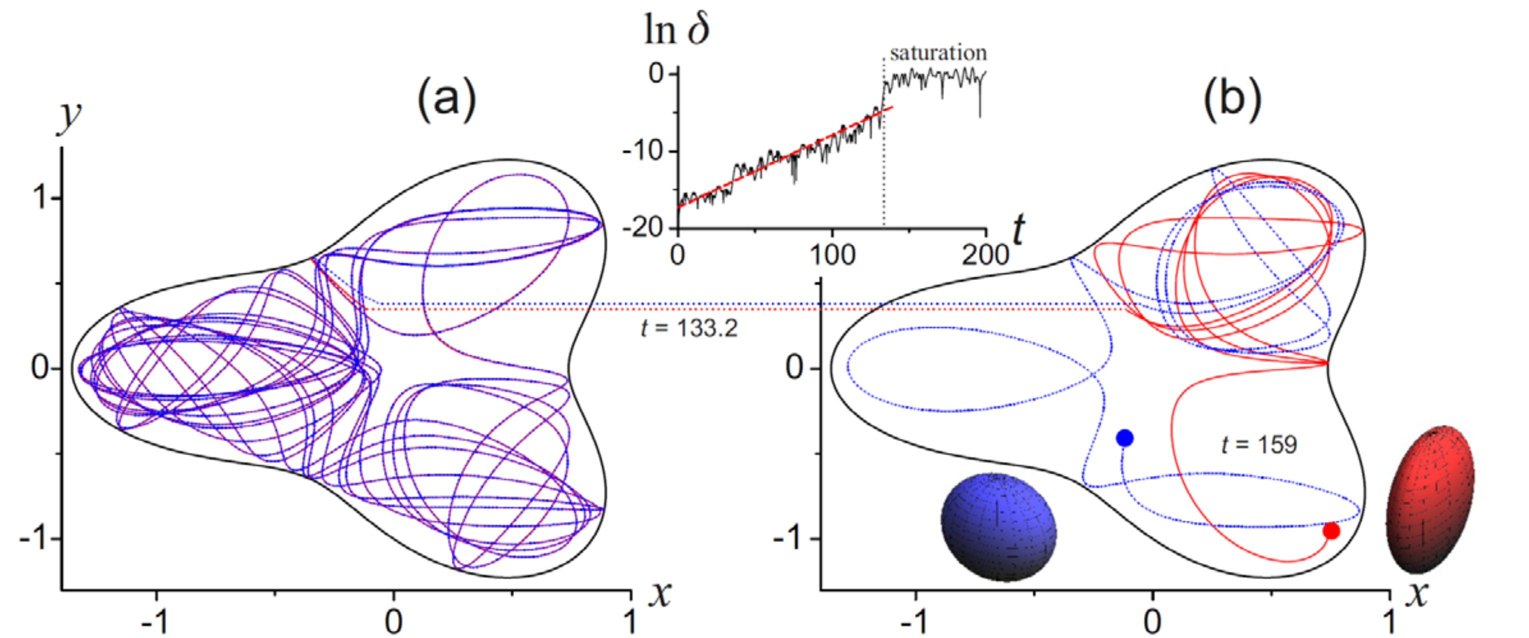


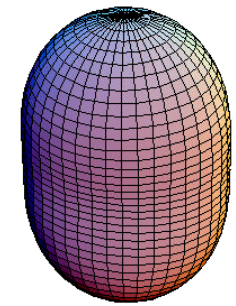
$\Rightarrow$ **complex structures in the phase space**
$\Rightarrow$ **exponential sensitivity to initial conditions**
$\Rightarrow$ **the system is not solvable**

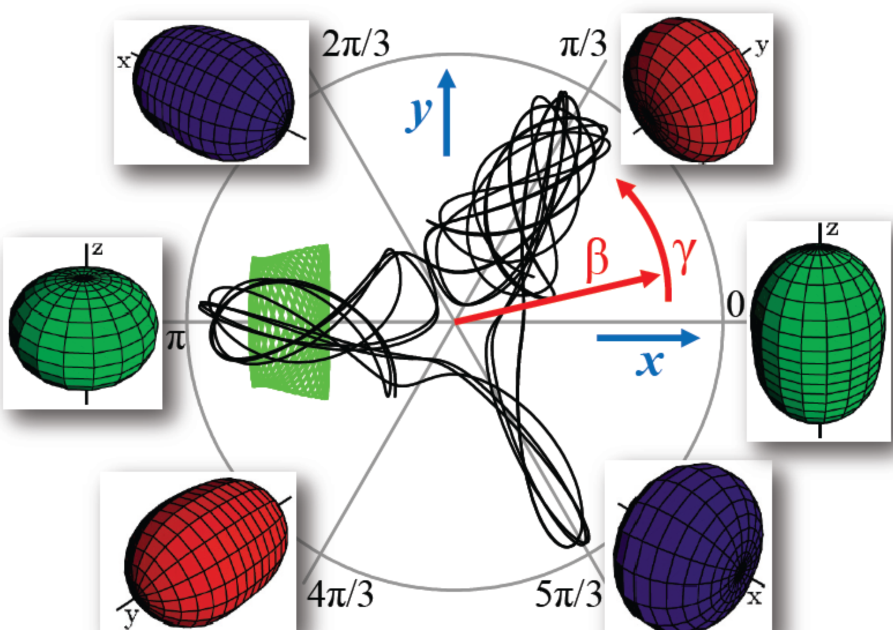
$\Rightarrow$ **Chaos**

Nuclear collective models

Stránský, Macek, Cejnar, NP News (2011)
Cejnar, Stránský, AIP Proc. (2014)







Quadrupole shape variables describing collective vibrations of nuclei

Collective models (including rotations):

GCM: **Geometric Collective Model**
Aage Bohr (1952)
Gneuss, Greiner *et al.* (1969)

IBM: **Interacting Boson Model**
Arima, Iachello (1975)

GCM

Aage Bohr (1952)
Gneuss, Greiner *et al.* (1969)

$$H=\underbrace{\sum_{\bullet=x,y,z}\frac{J'^{2}_{\bullet}}{2\mathfrak{I}_{\bullet}(\beta,\gamma)}}_{T_{\text{rot}}}+\underbrace{\frac{1}{2M}\left(\pi_\beta^2+\frac{\pi_\gamma^2}{\beta^2}\right)+\ldots}_{T_{\text{vib}}=\frac{1}{2M}\left(\pi_x^2+\pi_y^2\right)}+\underbrace{A\beta^2+B\beta^3\cos 3\gamma+C\beta^4+\ldots}_{V=A(x^2+y^2)+B(x^3-3y^2x)+\underbrace{C}_{>0}(x^2+y^2)^2}$$

(the T_{rot} term is crossed out)

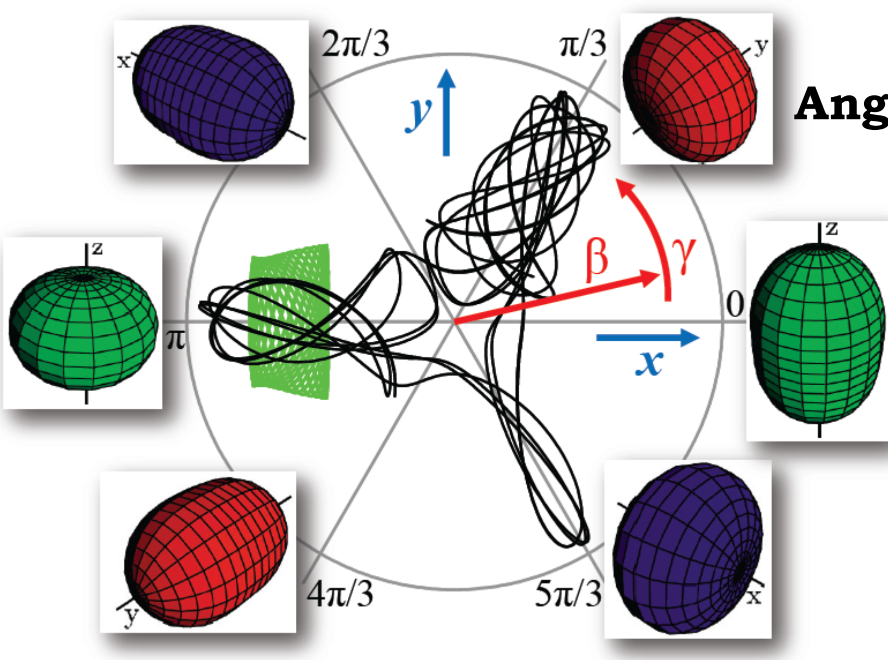
Hamiltonian

$$H=\frac{\sqrt{5}}{2M}[\pi\times\pi]^{(0)}+\overbrace{\ldots\ldots}^{\text{neglect}}+\sqrt{5}A[\alpha\times\alpha]^{(0)}-\sqrt{\frac{35}{2}}B\left[[\alpha\times\alpha]^{(2)}\times\alpha\right]^{(0)}+5C\left([\alpha\times\alpha]^{(0)}\right)^2+\overbrace{\ldots\ldots}^{\text{neglect}}$$

quadrupole tensor of collective coordinates
(2 shape + 3 Euler angles = 5D)

e.g:
$$\alpha_\mu^{(2)}\propto\int d\vec{r}\,\rho(\vec{r})\,r^2Y_{2\mu}(\vartheta,\varphi)$$
$$\alpha_{-\mu}^{(2)}=(-1)^\mu\alpha_{+\mu}^{(2)*}$$

…the corresponding momentum tensor



Angular momentum $J_\mu=-i\sqrt{10}[\alpha\times\pi^*]^{(1)}_\mu$ **= 0**

$\Rightarrow$ effectively **2D system**

Relevant coordinates obtained in the Principal Axes System defined by the diagonalization of α_{ij}

Shape variables
convenient parametrization of α_{ij} eigenvalues

$$\alpha_1\equiv\sqrt{\tfrac{5}{4\pi}}\beta\cos\left(\gamma-\tfrac{2\pi}{3}\right)$$
$$\alpha_2\equiv\sqrt{\tfrac{5}{4\pi}}\beta\cos\gamma$$
$$\alpha_3\equiv\sqrt{\tfrac{5}{4\pi}}\beta\cos\left(\gamma+\tfrac{2\pi}{3}\right)$$

GCM

Aage Bohr (1952)
Gneuss, Greiner *et al.* (1969)

$$H = \underbrace{\sum_{\bullet=x,y,z} \frac{J_\bullet'^2}{2\Im_\bullet(\beta,\gamma)}}_{T_{\rm rot}} + \underbrace{\frac{1}{2M}\left(\pi_\beta^2 + \frac{\pi_\gamma^2}{\beta^2}\right) + \ldots}_{T_{\rm vib} = \frac{1}{2M}\left(\pi_x^2 + \pi_y^2\right)} + \underbrace{A\beta^2 + B\beta^3 \cos 3\gamma + C\beta^4 + \ldots}_{V = A(x^2+y^2) + B(x^3-3y^2x) + \underbrace{C}_{>0}(x^2+y^2)^2}$$

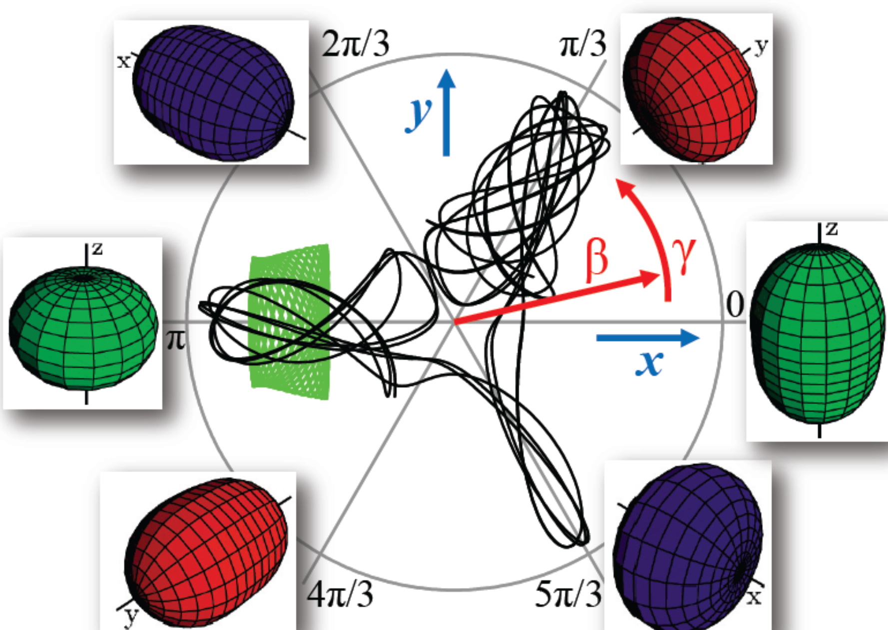
(the rotational term $T_{\rm rot}$ is crossed out)

Relation to the Hénon-Heiles system

Hénon, Heiles, Astron. J. 69, 73 (1964)

$$V_{\rm HH} = \tfrac{1}{2}(x^2+y^2) - \tfrac{1}{3}(x^3 - 3y^2x)$$

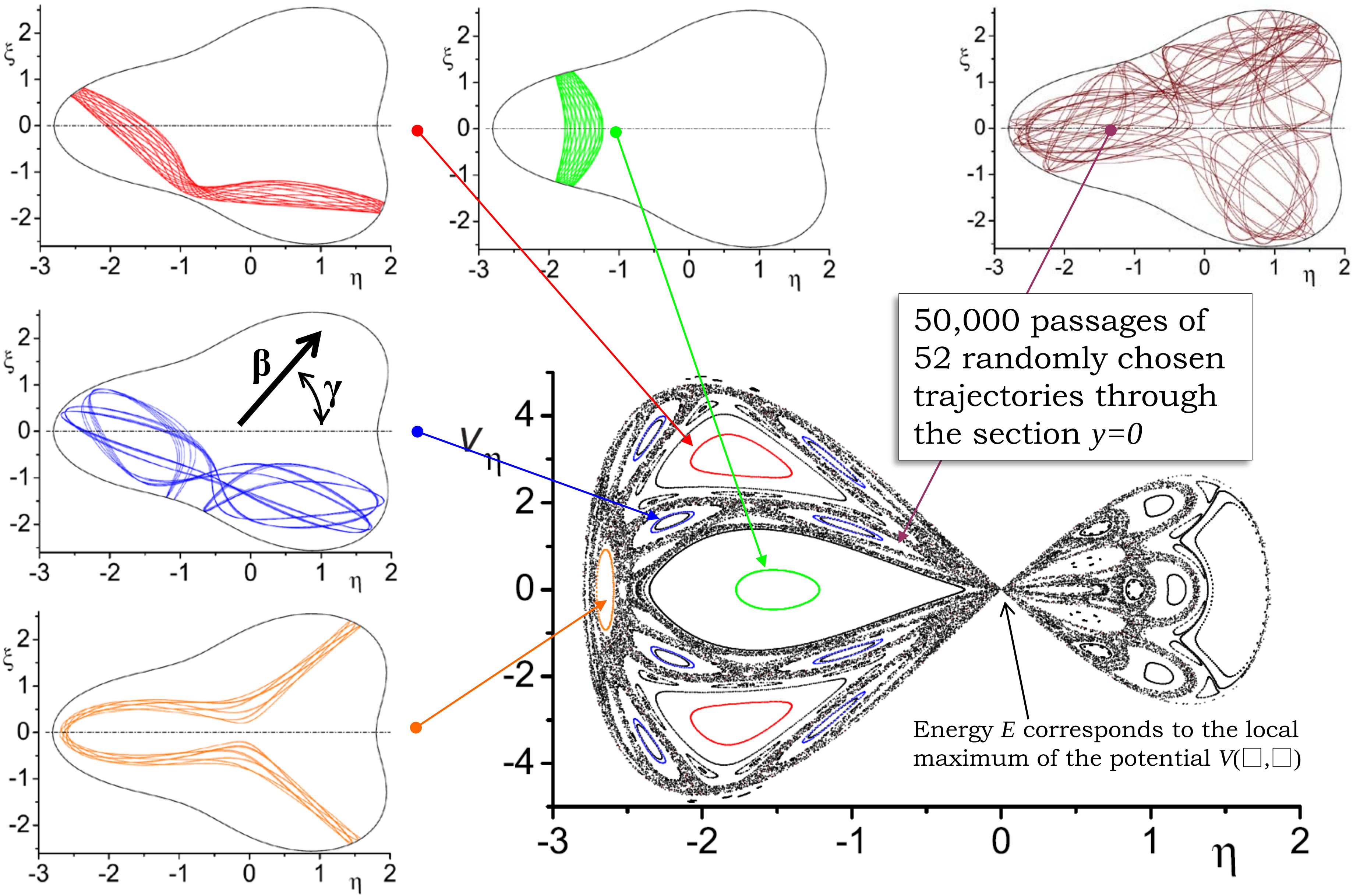
Paradigm of chaos !!! Model *inspired by* **motions of stars in an axially symmetric galactic potential**. The cylindrical variables substituted: $(z,r)\to(x,y)$. The HH potential is a toy example demonstrating complexity of the problem.



Poincaré maps

P.Cejnar, P.Stránský, PRL 93 (2004) 102502
P.Stránský, M.Kurian, P.Cejnar, PRC 74 (2006) 014306
P.Stránský, P.Cejnar, M.Macek, Ph.At.Nucl. 70 (2007) 1572

GCM examples



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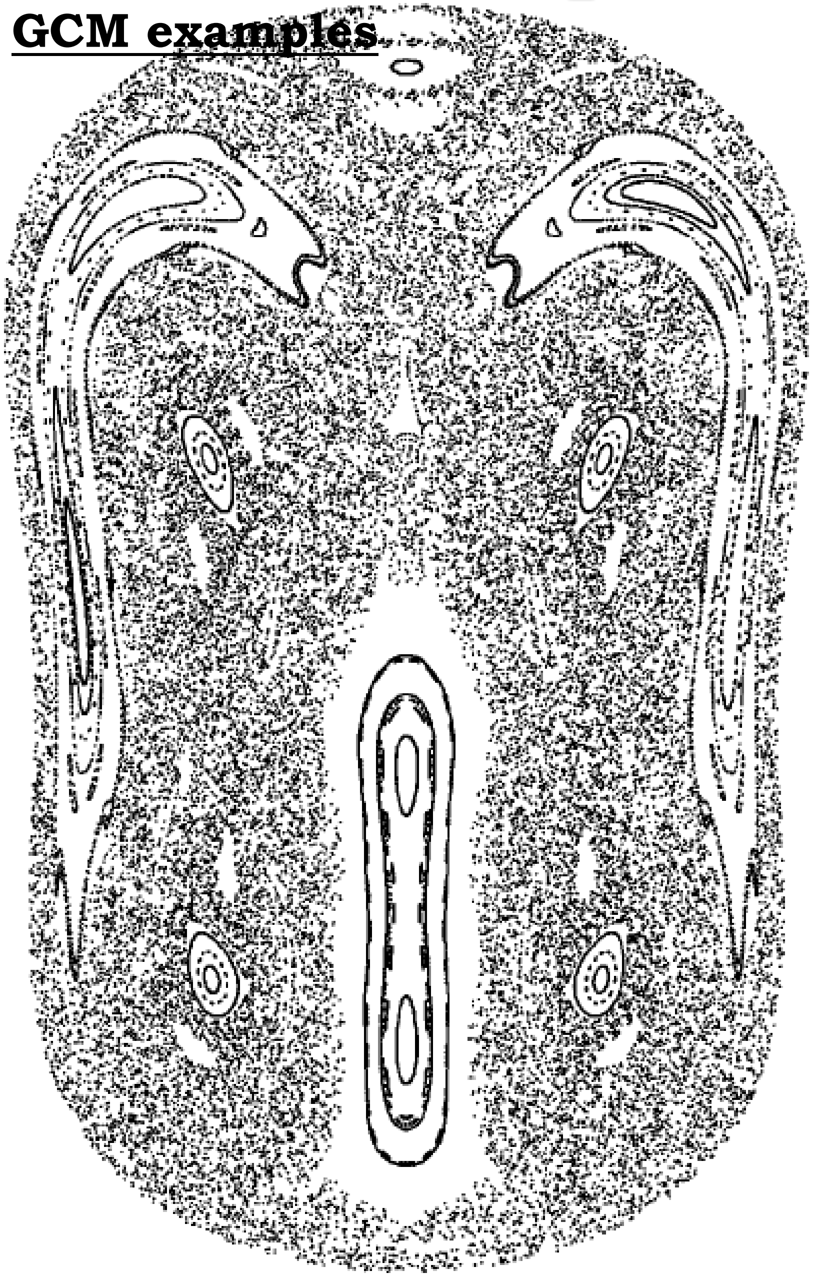
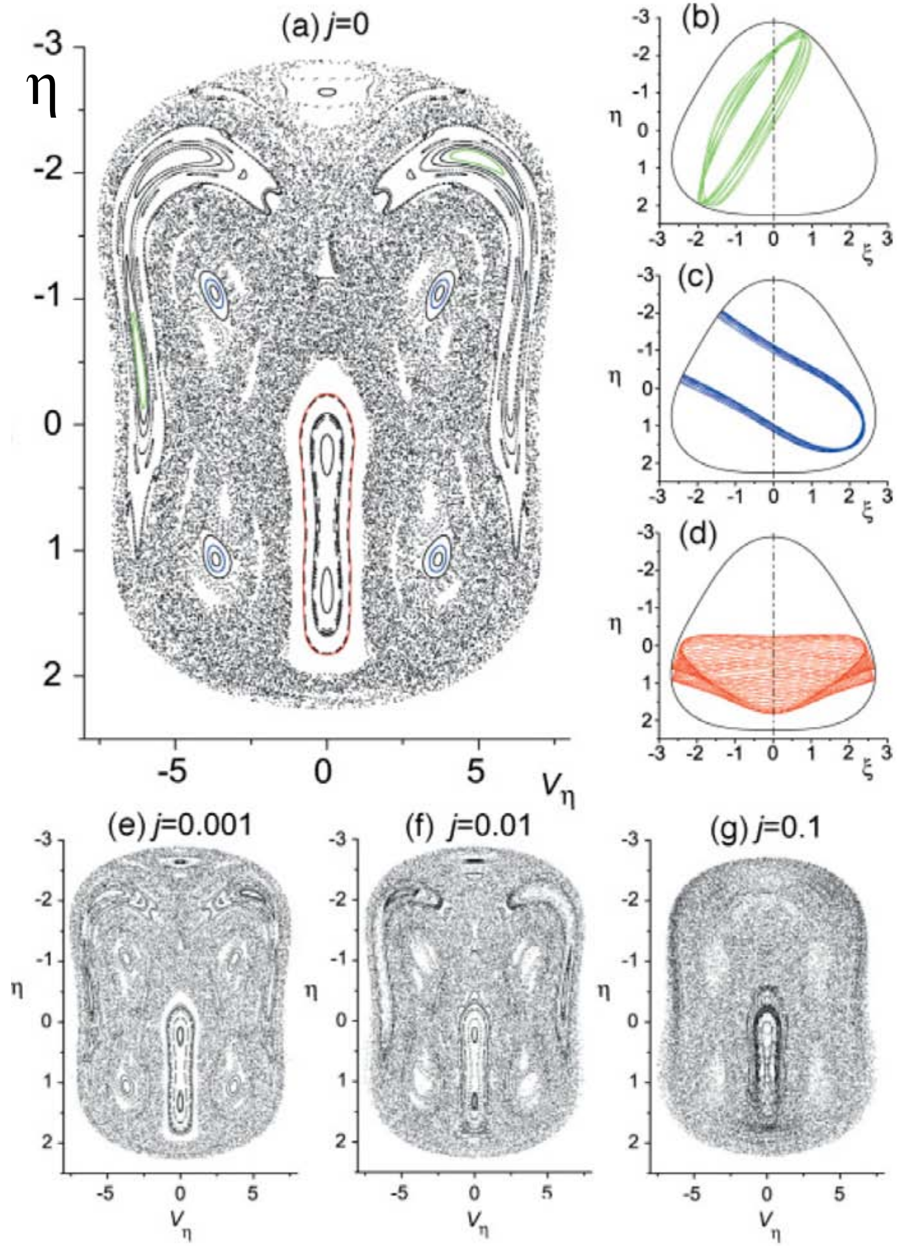
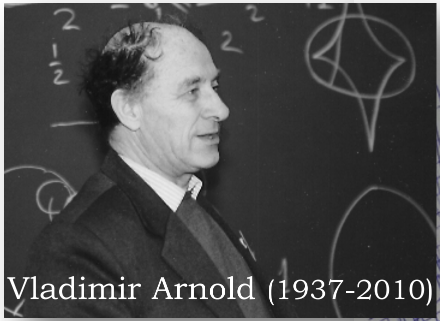
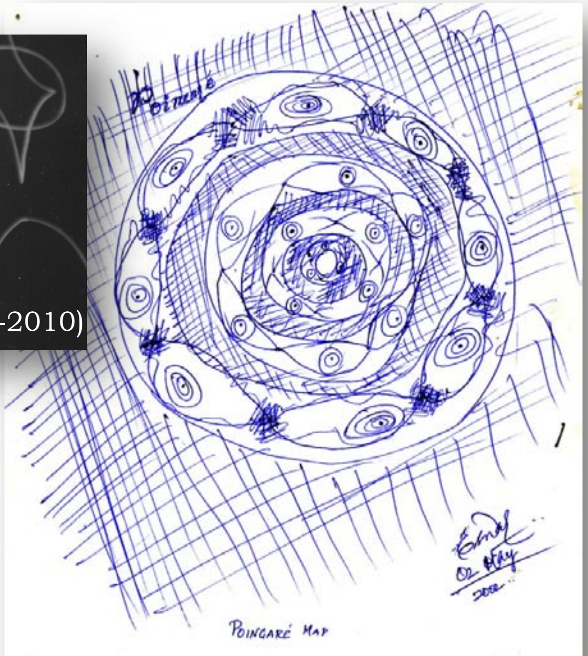
GCM examples





Poincaré maps





Scenario for the onset of chaos:

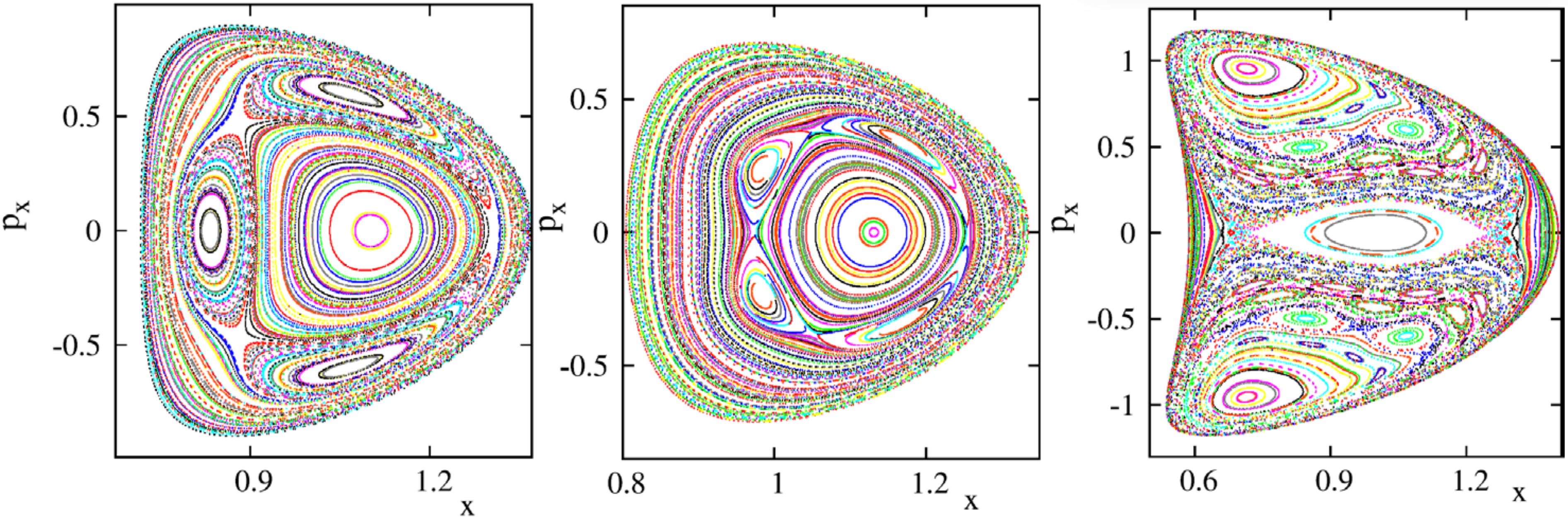
Poincaré-Birkhoff
1912, 1913, 1935

Kolmogorov-Arnold-Moser (KAM)
1954, 1963, 1962

Condition for the survival of an integrable torus in a perturbed system:

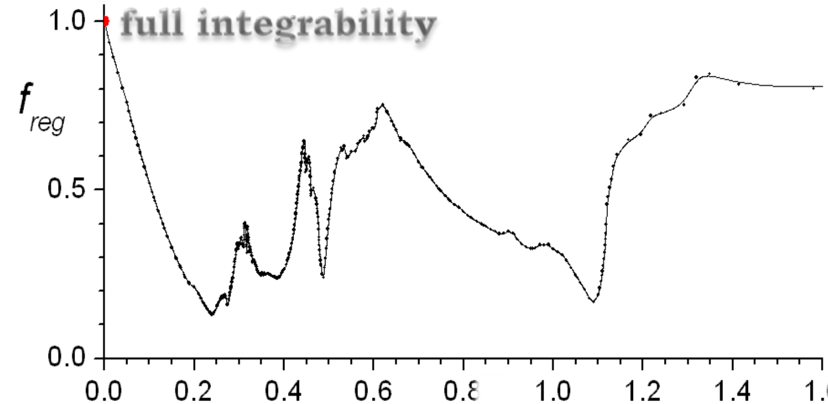
$$\mu \equiv \frac{\omega_2}{\omega_1} \qquad \left|\mu - \frac{m_1}{m_2}\right| > \frac{\text{const}}{|m_2|^{2+\varepsilon}} \quad \forall m_1, m_2 = 1,2,\cdots$$

IBM examples (courtesy of M.Macek)



Classical measure of chaos

P.Cejnar, P.Stránský, PRL 93 (2004) 102502
P.Stránský, M.Kurian, P.Cejnar, PRC 74 (2006) 014306



Regular phase-space fraction

$$f_{\text{reg}}(E) = \frac{\Omega_{\text{reg}}(E)}{\Omega_{\text{tot}}(E)}$$

$\Omega_{\text{reg}}(E)$
„Surface“ of the regular part (reg.orbits) of selected energy shell in phase space

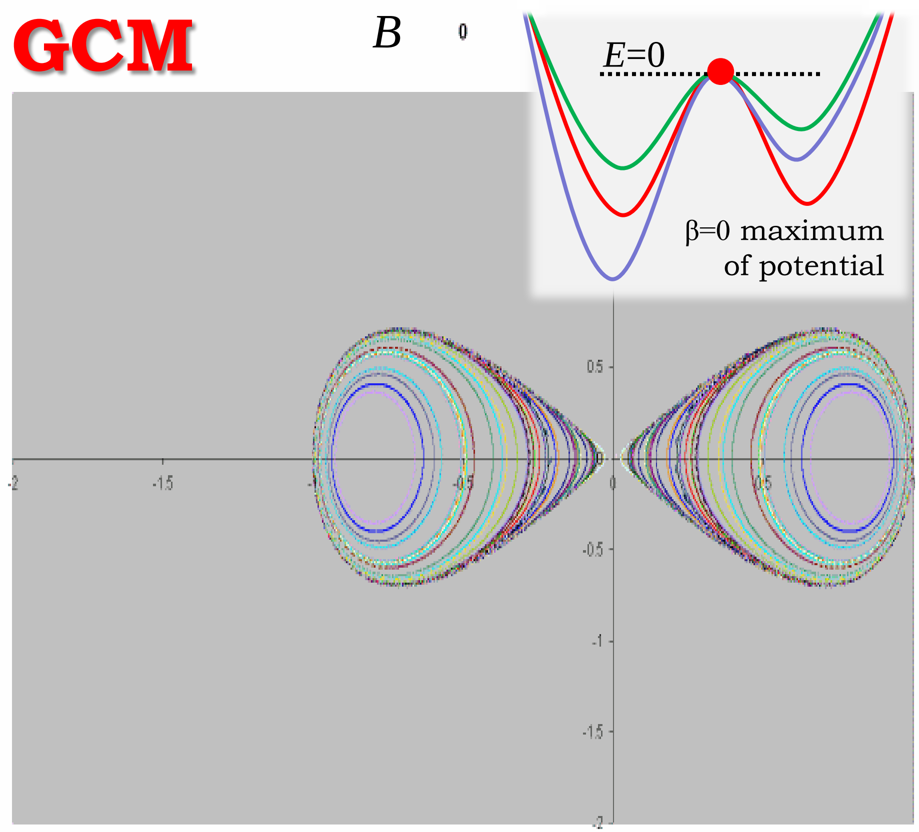
$\equiv$ ———————————

Total „surface“ of the energy shell:
$$\Omega_{\text{tot}}(E) \equiv \int \delta\big(E - H(\vec{p},\vec{q})\big)\, d\vec{p}\, d\vec{q}$$

$$\in [0,1]$$

[chaotic … transitional … regular]

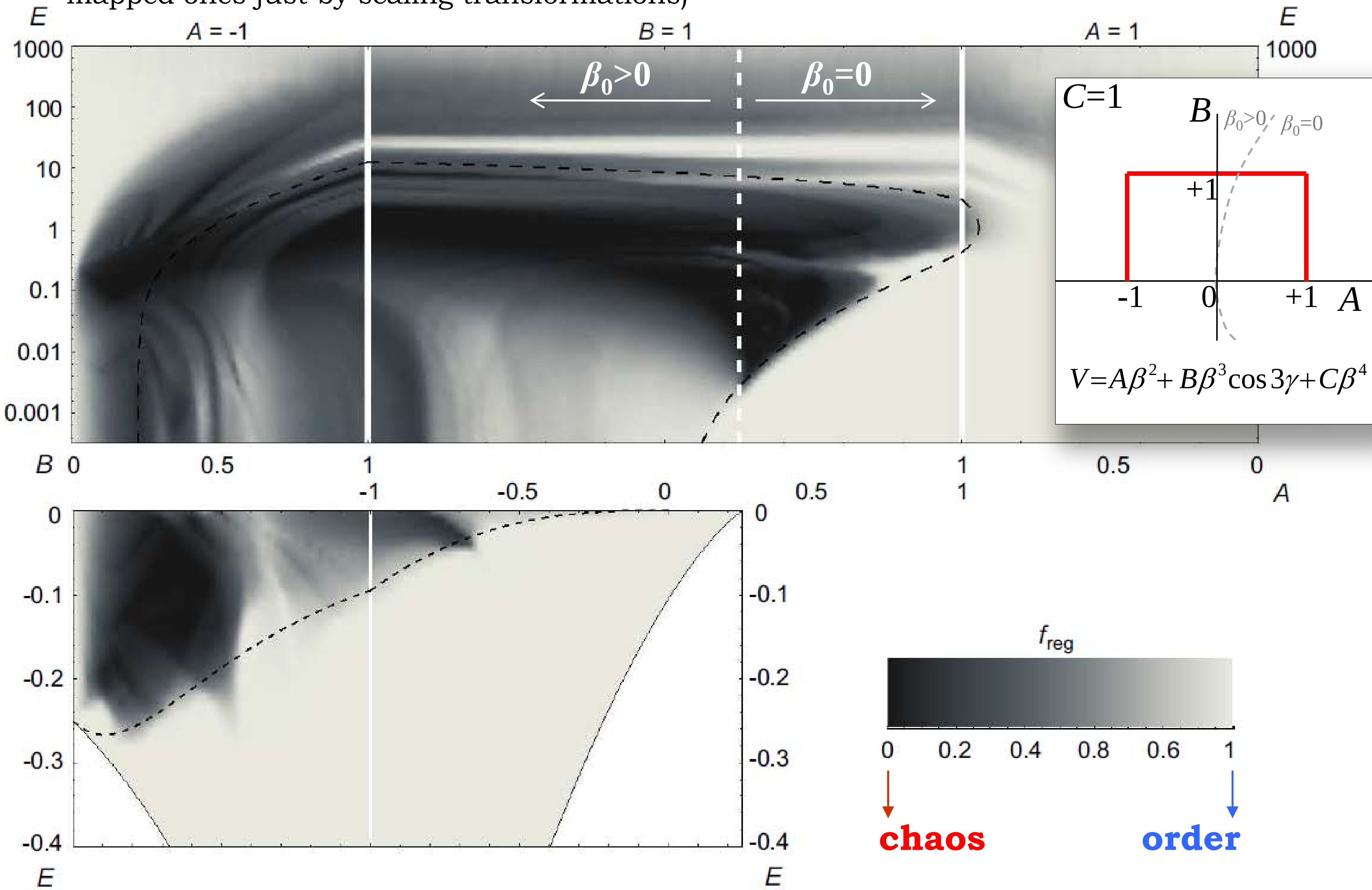
GCM



GCM map of chaos

Stránský, Macek, Cejnar, NP News (2011)

(other parameter choices differ from the mapped ones just by scaling transformations)



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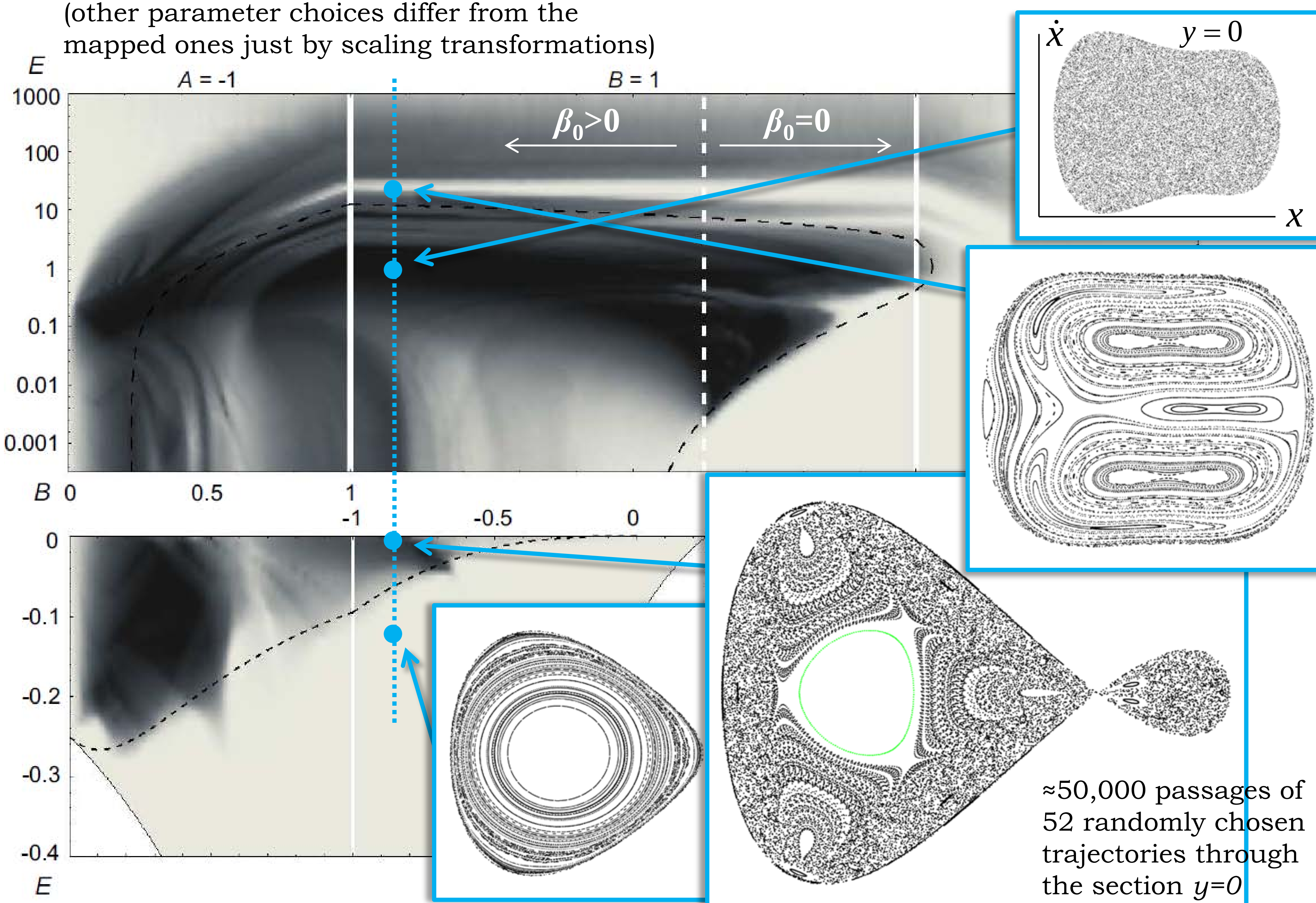

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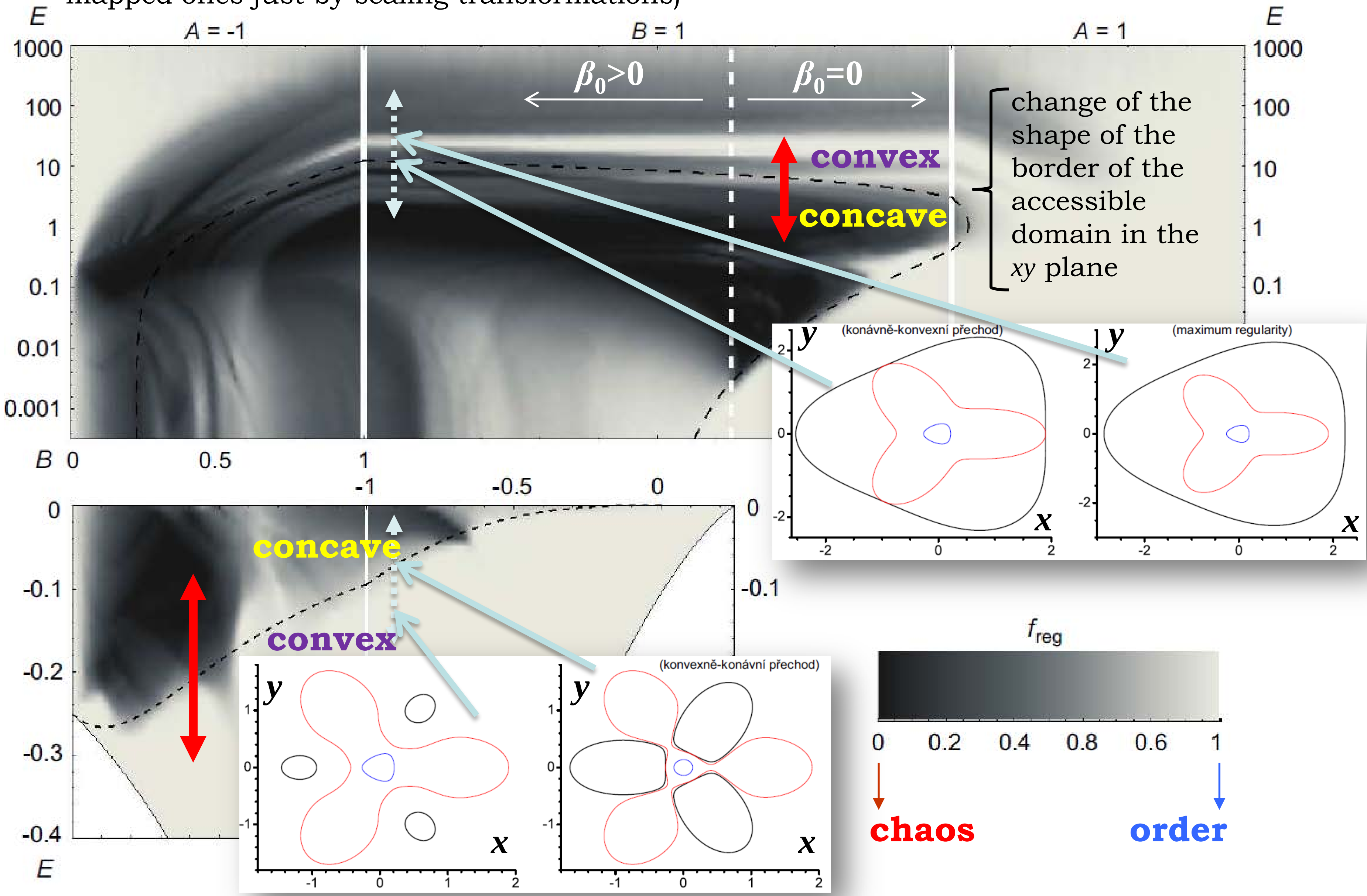


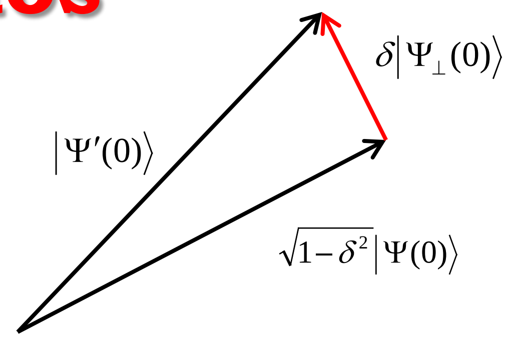
Puzzle of Quantum Chaos

1) <u>Linearity</u> of QM

$$|\Psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\Psi(0)\rangle$$

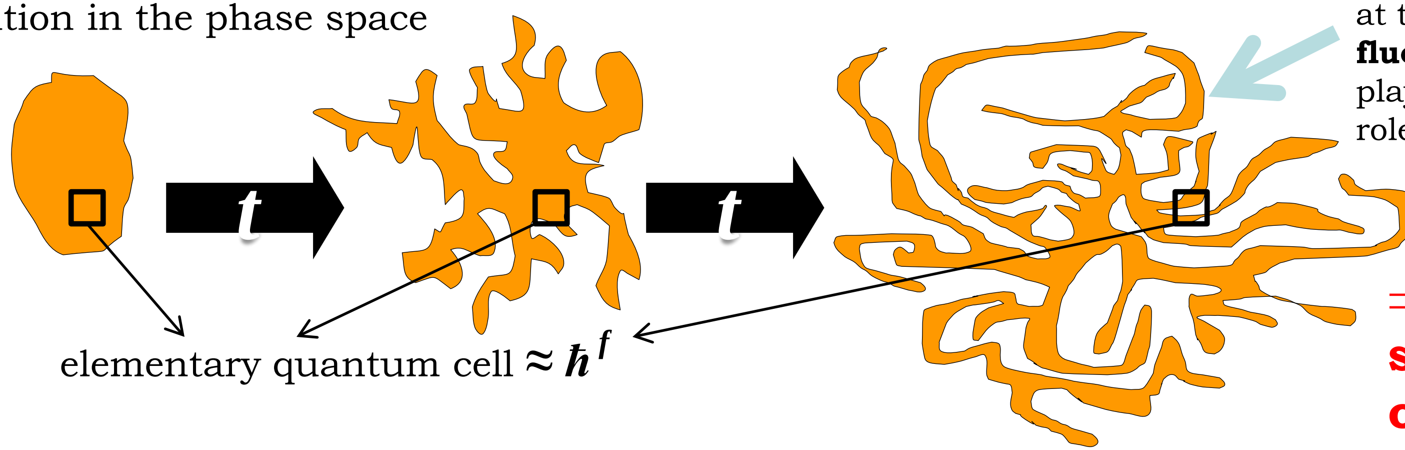
$$|\Psi'(t)\rangle = \sqrt{1-\delta^2}\,|\Psi(t)\rangle + \delta|\Psi_\perp(t)\rangle$$

$\Rightarrow$ the Hilbert-space distance of both solutions remains the same for all times => no butterfly wing effect



2) <u>Long-time</u> evolution of a classical chaotic system

Evolution in the phase space

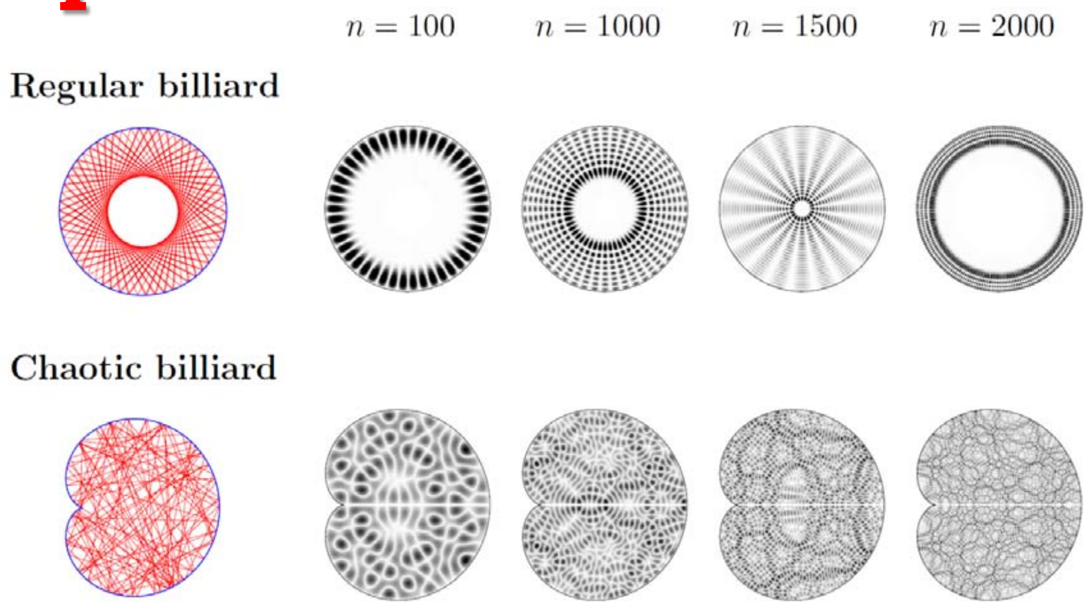


at this stage **quantum fluctuations** start playing an important role => classical description no more applicable

$\Rightarrow$ **quantum suppression of chaos**

<u>Problem 2)</u> can be “solved” by considering the *interaction of the system with an environment.* <u>Problem 1)</u> can be bypassed by declaring that quantum chaos is not a “phenomenon” but rather a “branch of physics” studying *quantum properties of the classically chaotic systems.*

Spectral correlations



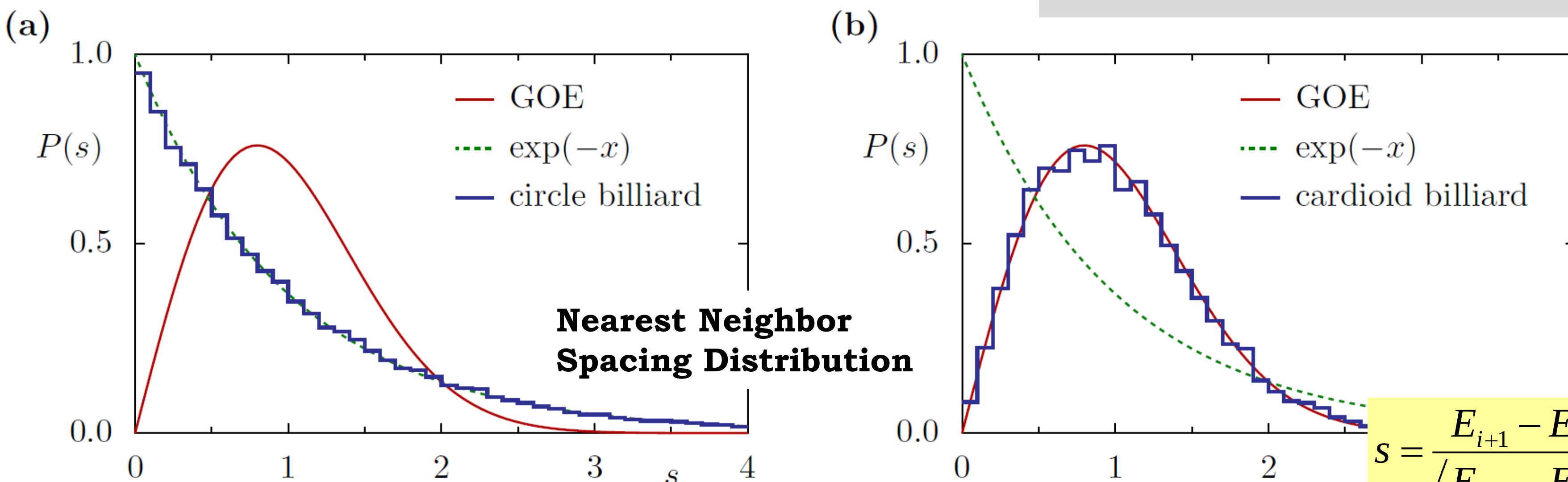
From: A. Bäcker, Computing in Science and Engineering 9 (2007)

"Bohigas conjecture"

[O. Bohigas, M.J. Giannoni, C. Schmit, PRL 52, 1 (1984)]

Quantum chaotic systems with discrete energy spectra (i.e. bound systems) exhibit strong spectral correlations of the same type as the spectra of random Hamiltonians (this holds only for subsets of levels with the same conserved quantum numbers)

Quantum regular systems have uncorrelated (Poissonian) spectra



Nearest Neighbor Spacing Distribution

$$s = \frac{E_{i+1} - E_i}{\langle E_{i+1} - E_i \rangle}$$

Short range spectral correlations

regular

$$P_{\mathrm{NNS}} = e^{-s}$$

□ = 0

Poisson

$$P_{\mathrm{NNS}} = N_\omega\, s^\omega e^{-\alpha_\omega s^{\omega+1}}$$

$N_\omega = (\omega+1)\alpha_\omega$ $\alpha_\omega = \Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1}$

transitional systems

Brody

chaotic

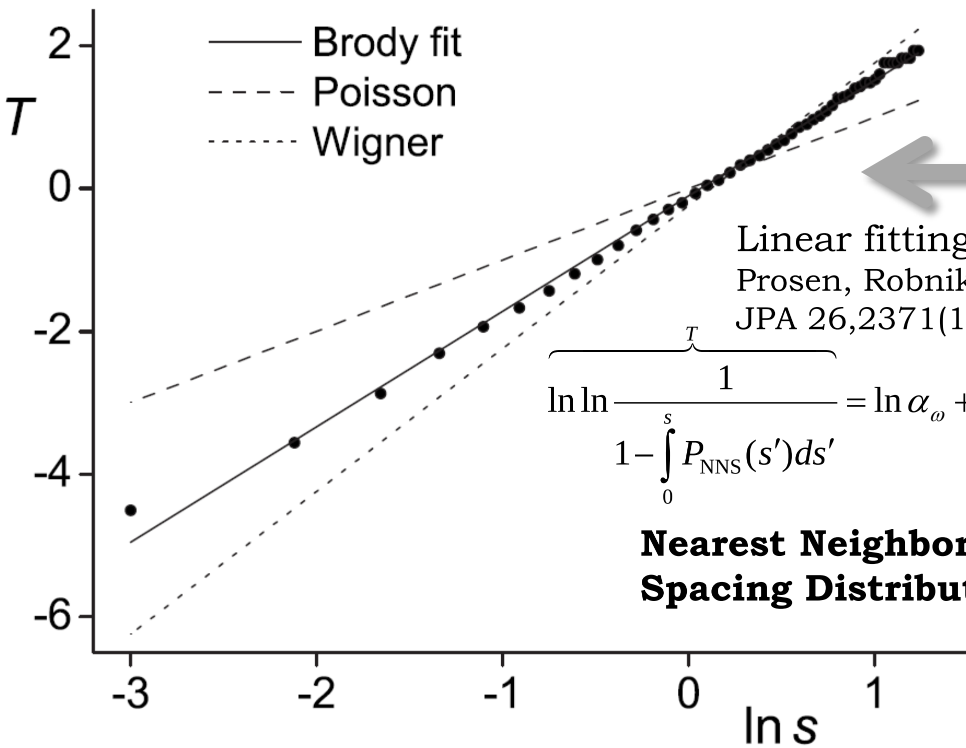
$$P_{\mathrm{NNS}} = \frac{\pi}{2} s\, e^{-\frac{\pi}{4}s^2}$$

□ = 1

Wigner (~GOE)

GCM example: fit of data from a narrow energy interval:

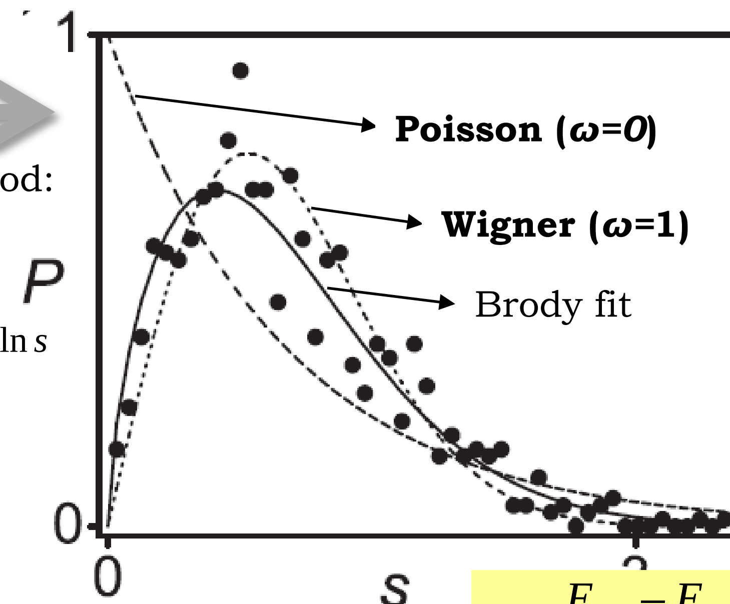
$$1-\omega \leftrightarrow f_{\mathrm{reg}}$$



Linear fitting method:
Prosen, Robnik,
JPA 26,2371(1993)

$$\overbrace{\ln\ln\frac{1}{1-\int_0^s P_{\mathrm{NNS}}(s')ds'}}^{T} = \ln\alpha_\omega + (1+\omega)\ln s$$

Nearest Neighbor Spacing Distribution



$$s = \frac{E_{i+1}-E_i}{\langle E_{i+1}-E_i\rangle}$$

Short range spectral correlations

regular

$$P_{\mathrm{NNS}} = \mathrm{e}^{-s}$$

□ = 0

Poisson

$$P_{\mathrm{NNS}} = N_{\omega}\, s^{\omega} \mathrm{e}^{-\alpha_{\omega} s^{\omega+1}}$$

$N_{\omega} = (\omega+1)\alpha_{\omega}$ $\quad$ $\alpha_{\omega} = \Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1}$

transitional systems

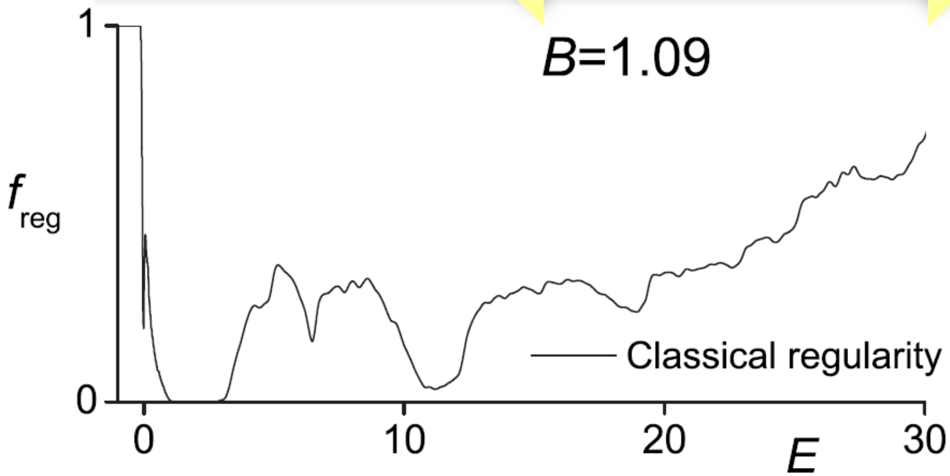
Brody

chaotic

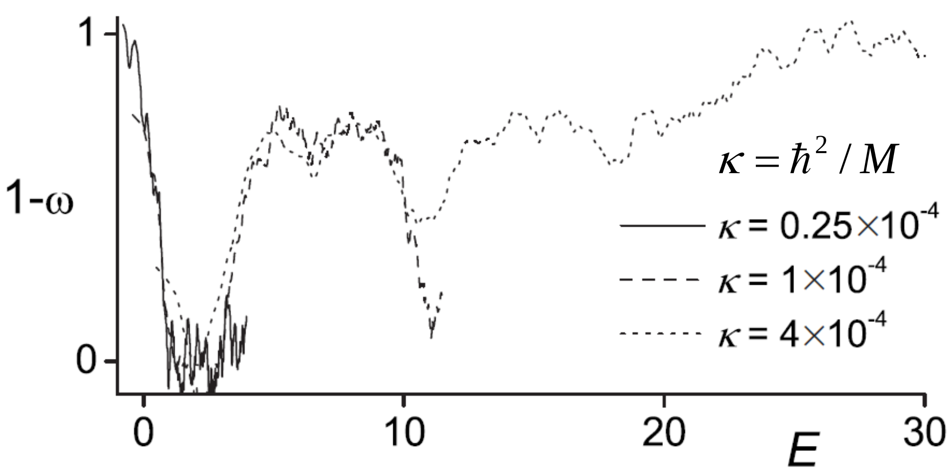
$$P_{\mathrm{NNS}} = \frac{\pi}{2}\, s\, \mathrm{e}^{-\frac{\pi}{4}s^2}$$

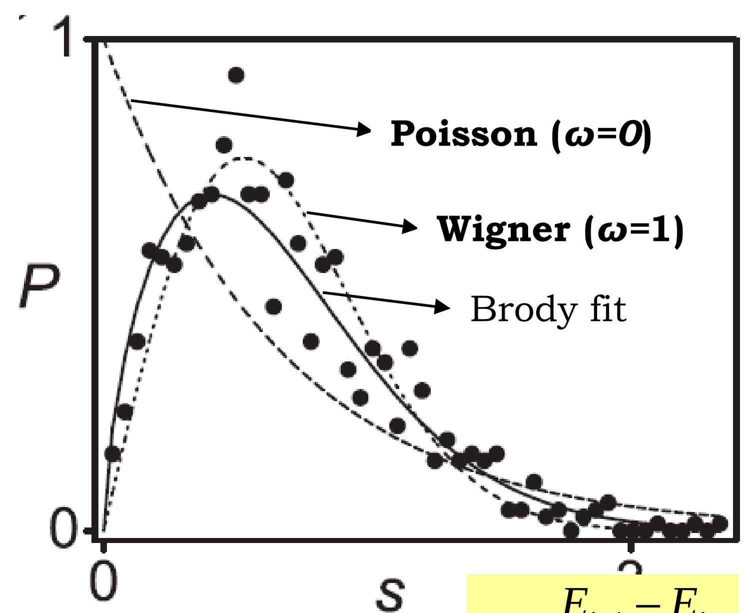
□ = 1

Wigner (~GOE)



$$1-\omega \leftrightarrow f_{\mathrm{reg}}$$





$$s = \frac{E_{i+1} - E_i}{\langle E_{i+1} - E_i \rangle}$$

Short range spectral correlations

regular

$$P_{\rm NNS} = e^{-s}$$

□ = 0

Poisson

$$P_{\rm NNS} = N_\omega \, s^\omega e^{-\alpha_\omega s^{\omega+1}}$$

$N_\omega = (\omega+1)\alpha_\omega$ $\alpha_\omega = \Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1}$

transitional systems

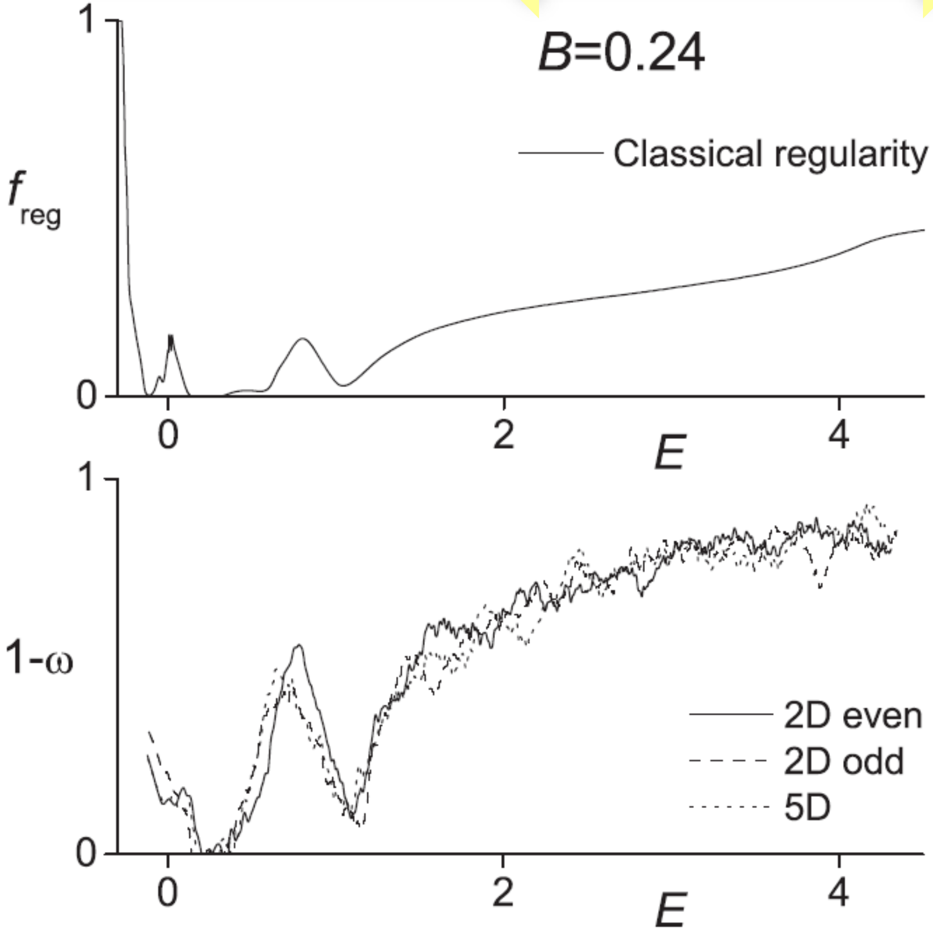
Brody

chaotic

$$P_{\rm NNS} = \frac{\pi}{2} s \, e^{-\frac{\pi}{4}s^2}$$

□ = 1

Wigner (~GOE)



$$1-\omega \leftrightarrow f_{\rm reg}$$

Quantization schemes:

Classical: $$T_{\rm vib} = \frac{1}{2M}\left(\pi_x^2 + \pi_y^2\right) = \frac{1}{2M}\left(\pi_\beta^2 + \frac{\pi_\gamma^2}{\beta^2}\right)$$

Quantum:

$$\hat{T}_{\rm vib}^{(2D)} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -\frac{\hbar^2}{2M}\left(\frac{1}{\beta}\frac{\partial}{\partial\beta}\beta\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{\partial^2}{\partial\gamma^2}\right)$$

$$\Psi(\beta,\gamma) = \Psi(\beta,\gamma + k\tfrac{2\pi}{3})$$

(a) 2D even
(b) 2D odd

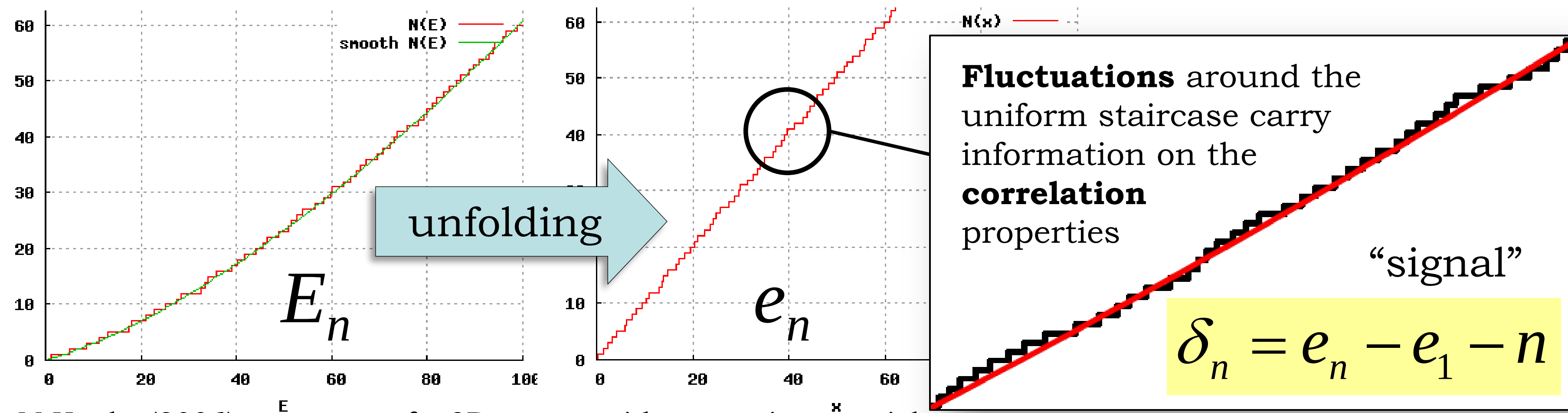
$$\Psi(\beta,\gamma) = \pm\Psi(\beta,-\gamma)$$

$$\hat{T}_{\rm vib}^{(5D)} = -\frac{\hbar^2}{2M}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}\right)$$

(c) 5D

Long range spectral correlations

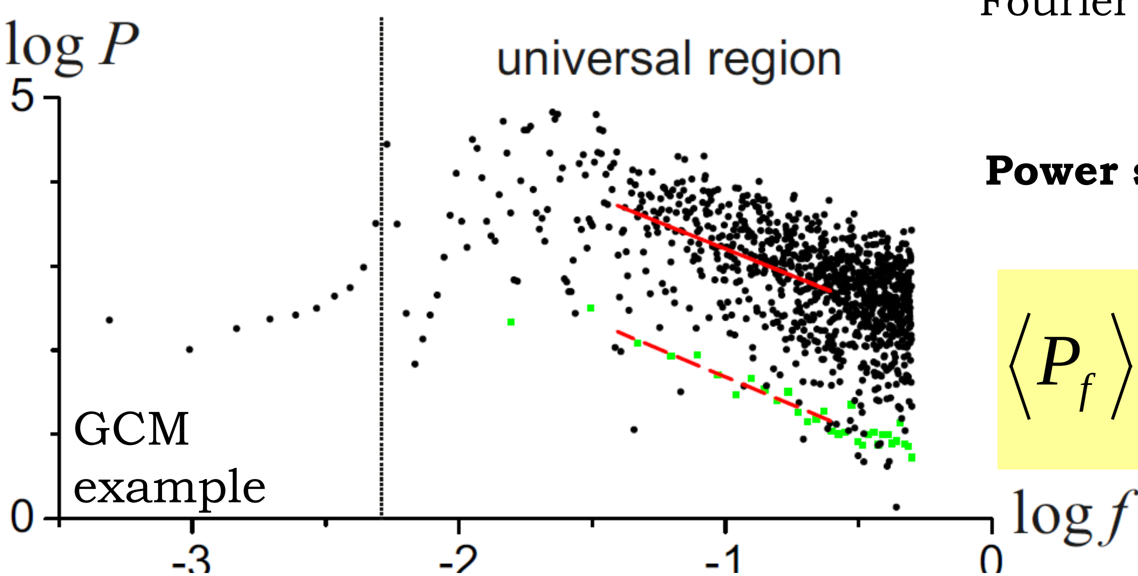
Original staircase (integrated) spectrum **Unfolded staircase spectrum** **Noise analysis**



M.Hanke (2006): spectrum of a 2D system with a quartic potential

Fourier spectrum $\tilde{\delta}_f = \frac{1}{M}\sum_{n=1}^{M} \delta_n e^{-i\frac{2\pi f n}{M}}$

Power spectrum $P_f = \left|\tilde{\delta}_f\right|^2$



$$\langle P_f \rangle = \frac{1}{f^{\alpha}}$$

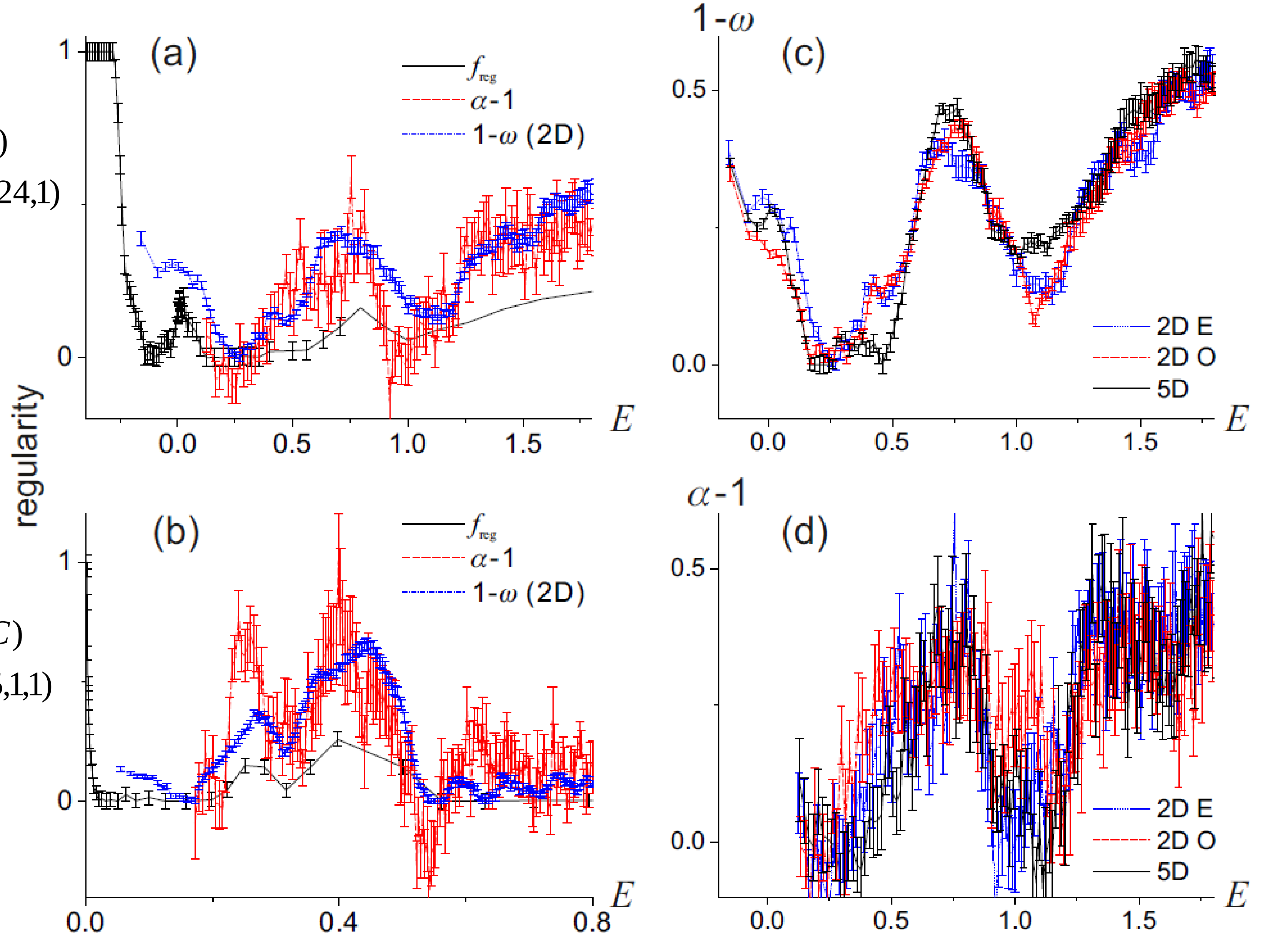
$$\alpha - 1 \leftrightarrow f_{\text{reg}}$$

Long range spectral correlations

Cejnar, Stránský, AIP Proc. (2014)

$(A, B, C) = (-1, 0.24, 1)$

$(A, B, C) = (0.25, 1, 1)$



Peres lattices

Asher Peres (1934-2005)

A visual method to detect chaos in systems with $f = 2$

Quantum **analog of Poincaré maps** in classical systems

[Peres, PRL 53, 1711 (1984)]

Integrals of motions for non-integrable systems:

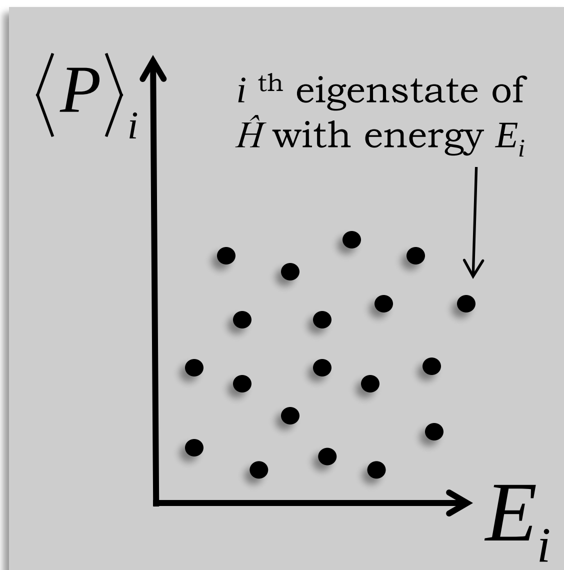
$$\overline{\hat{P}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \hat{P}_{\mathrm{H}}(t)\,dt$$

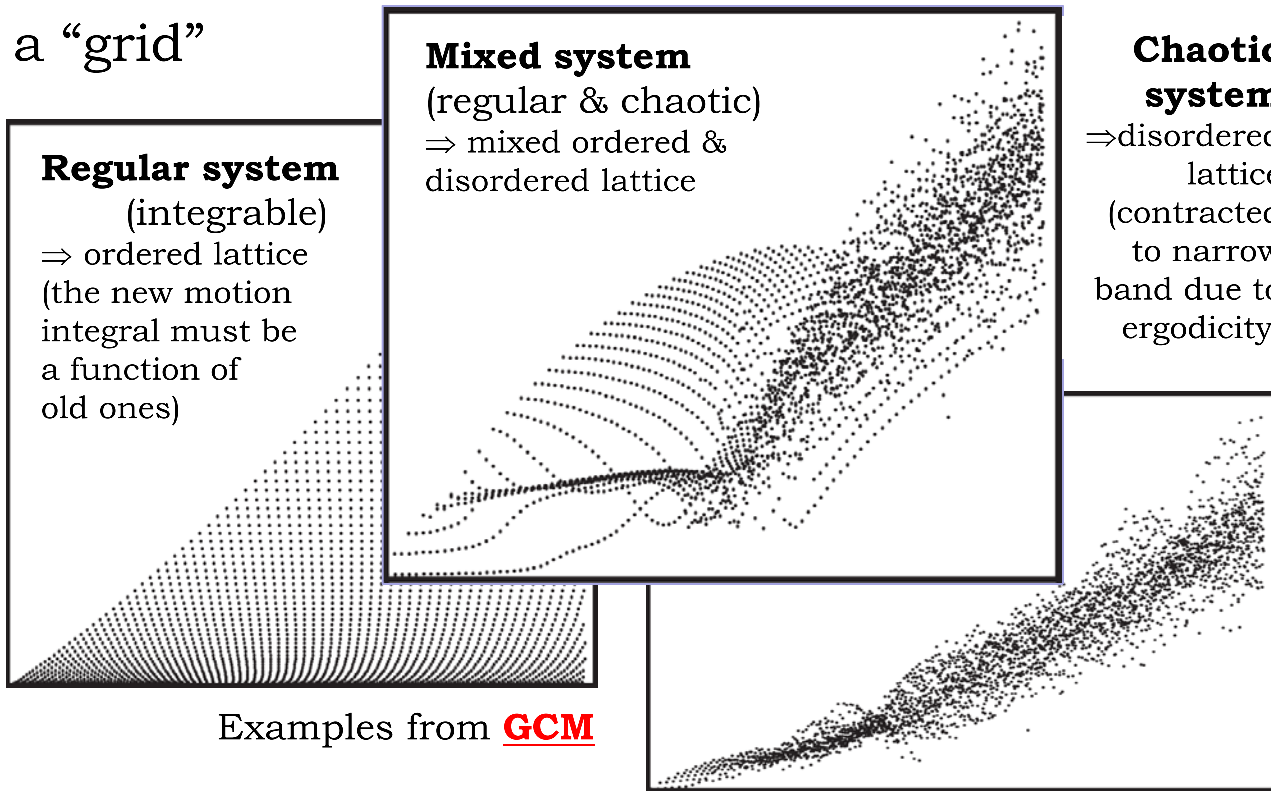
time average of an **arbitrary observable *P***

$$[\overline{\hat{P}}, \hat{H}] = 0 \qquad \overline{P}_i \equiv \langle \psi_i | \overline{\hat{P}} | \psi_i \rangle = \langle \psi_i | \hat{P} | \psi_i \rangle \equiv \langle P \rangle_i$$

time average — for stationary states — state average

Energy spectrum as a "grid"





Examples from **GCM**

Stránský, Hruška, Cejnar, PRE 79, 046202, 066201 (2009)

Peres lattices

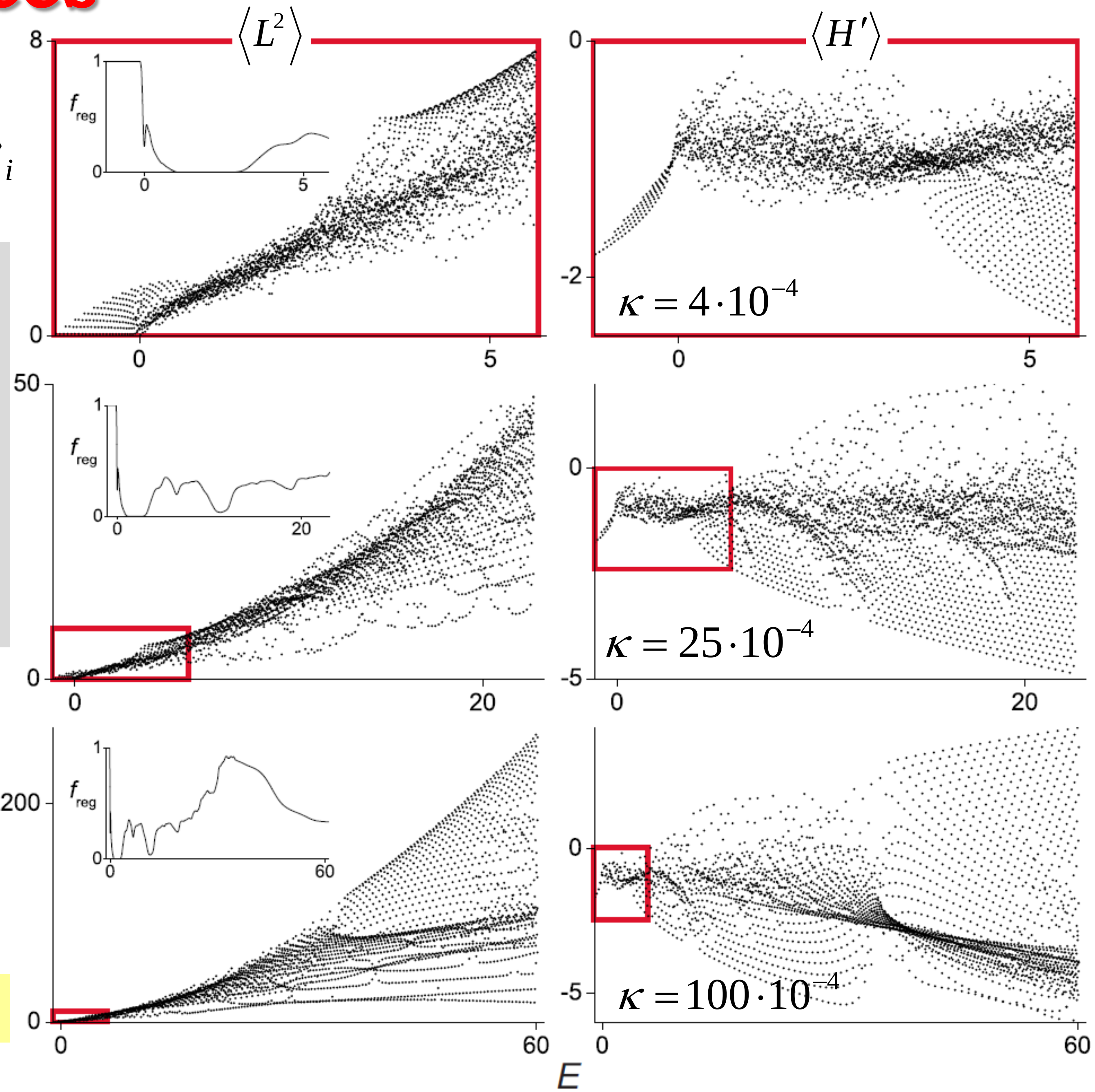
$$\langle \psi_i | \hat{P} | \psi_i \rangle \equiv \langle P \rangle_i$$

Left:

$$\hat{P} = -\hbar^2 \frac{\partial^2}{\partial \gamma^2} \equiv \hat{L}^2$$

Right:

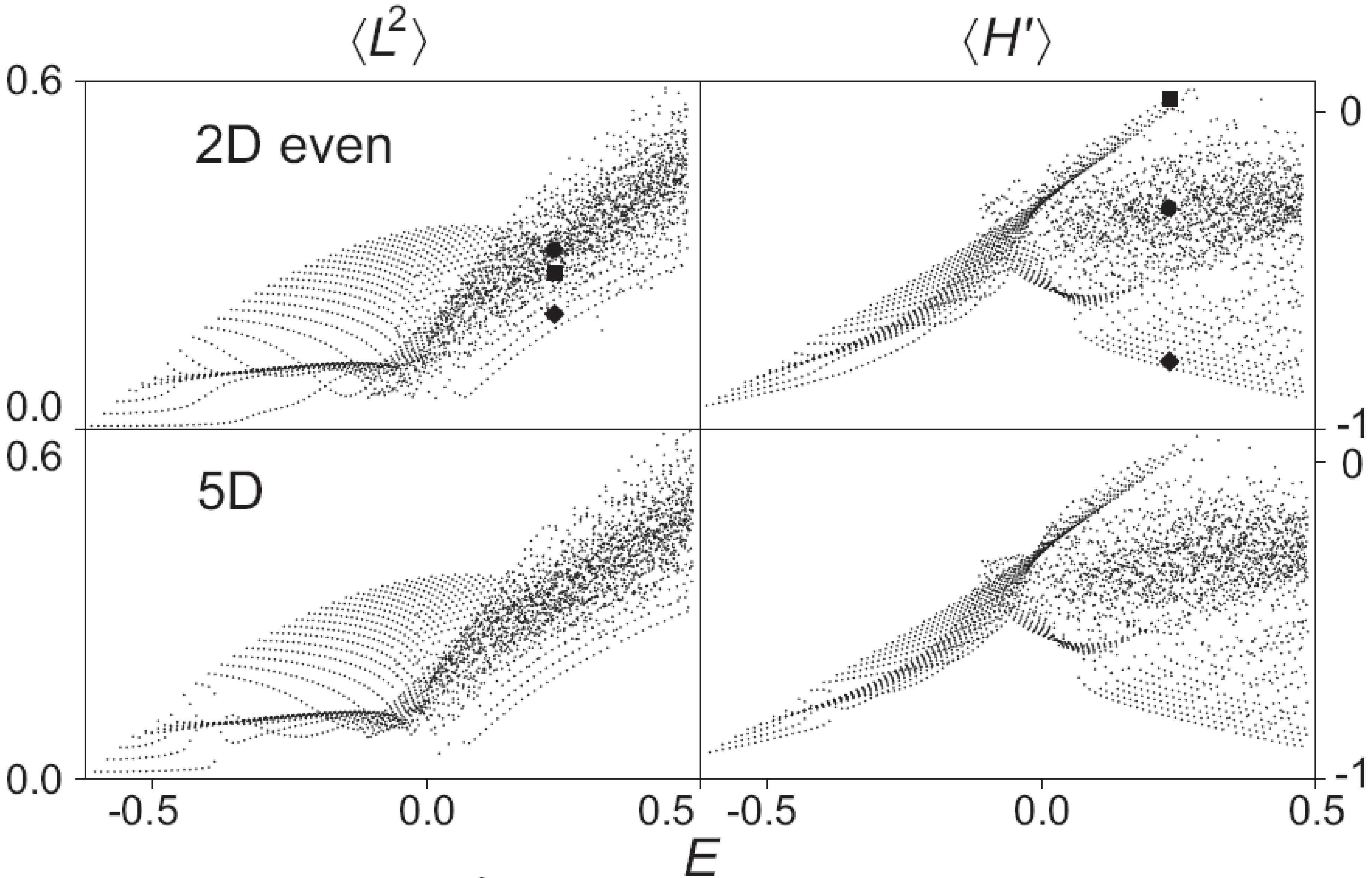
$$\hat{P} = \beta^3 \cos 3\gamma \equiv \hat{H}'$$



B=1.09 $\kappa = \hbar^2 / M$

Stránský, Hruška, Cejnar, PRE 79, 046202, 066201 (2009)

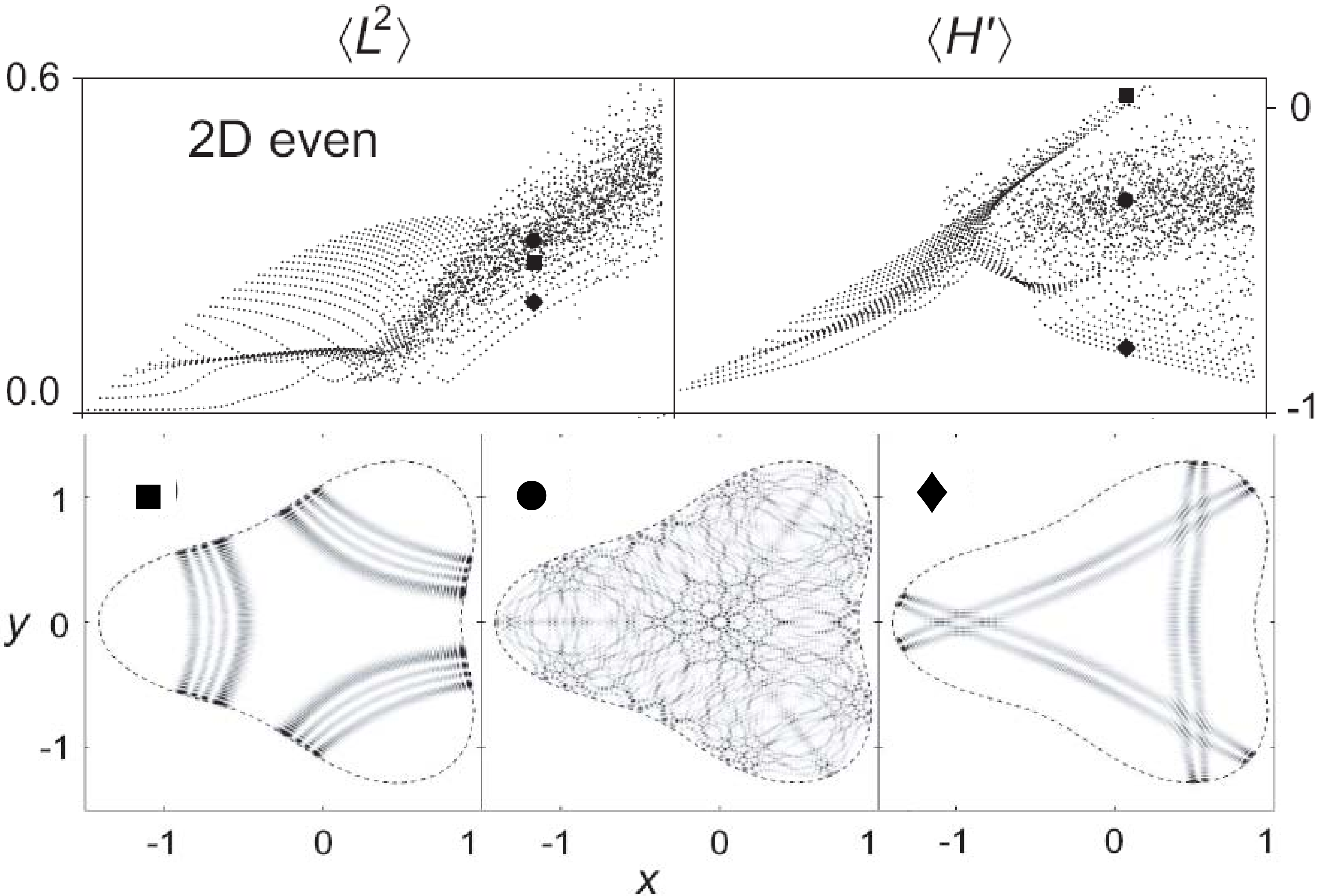
Peres lattices



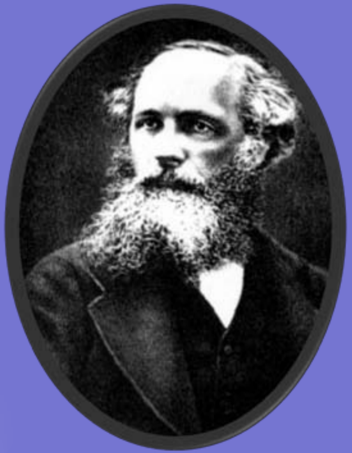
$B=0.62 \quad \kappa = 25 \cdot 10^{-6}$

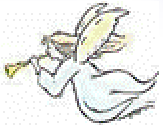
Stránský, Hruška, Cejnar, PRE 79, 046202, 066201 (2009)

Peres lattices



Encoding physics

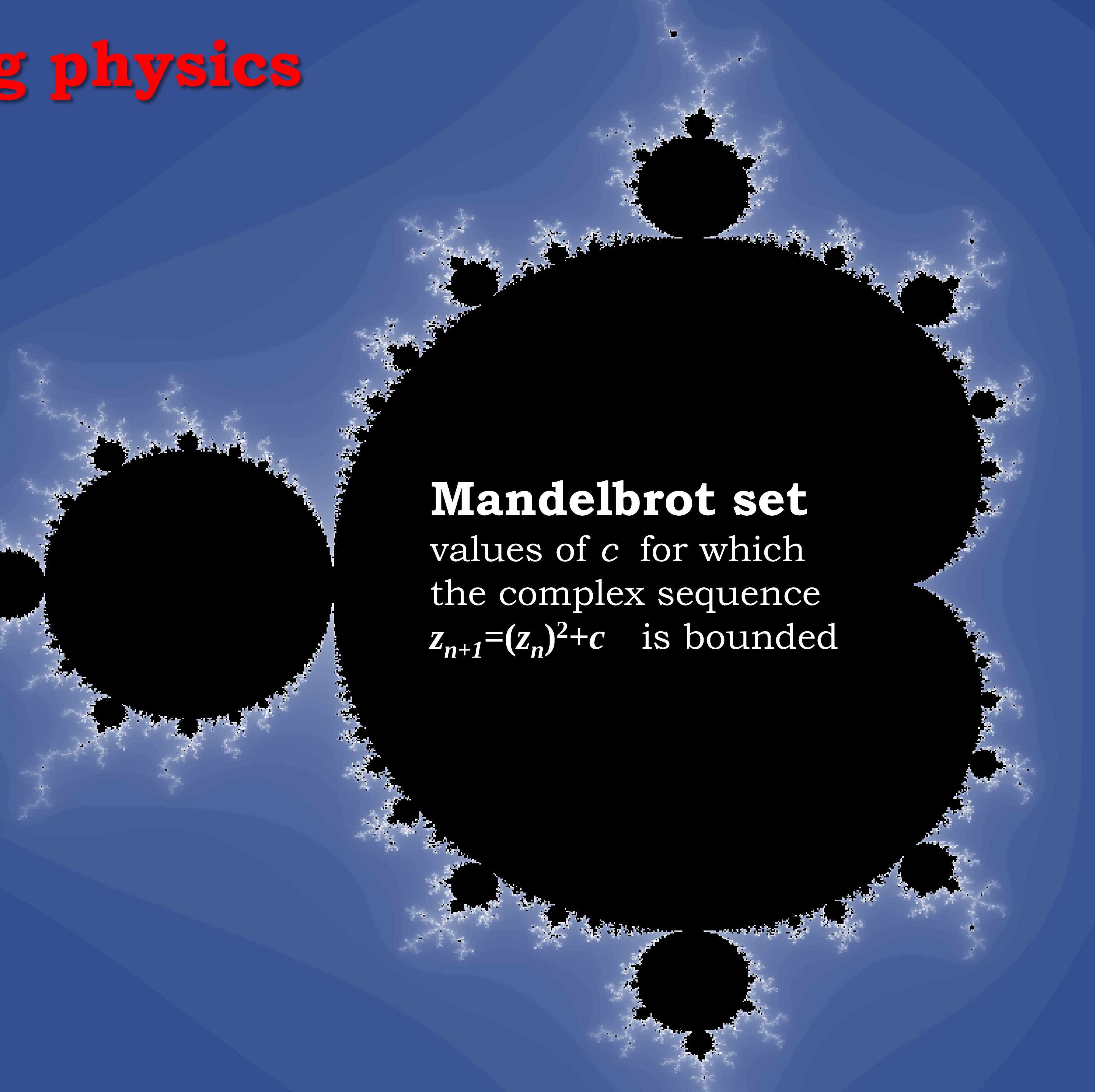




$$\operatorname{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \operatorname{div}\vec{D} = \rho$$

$$\operatorname{rot}\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \qquad \operatorname{div}\vec{B} = 0$$

Decoding physics



Mandelbrot set

values of c for which the complex sequence $z_{n+1}=(z_n)^2+c$ is bounded