α_s FROM HADRONIC τ DECAYS by NON-POWER PERTURBATION THEORY

- τ is the only known lepton heavy enough to decay into hadrons.
- The total hadronic width is inclusive, which allows an accurate calculation of the width.
- A very precise method to obtain small npt contributions to $\alpha_s(M_\tau^2)$ from hadronic τ decays. BraatenNarisonPich92,LeDibPich92,PDG
- IMPRESSIVE recent achievements in pt QCD: β and Adler function in massless QCD to 4 loops (good for processes measured at LHC). *WE PROPOSE* a new,
- NONPOWER series replacing the standard pt expansion in the α_s^n.
- This method can improve the methods of the determination of α_s from hadronic τ decays.

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α_s FROM HADRONIC τ DECAYS.
Non-power alternatives to divergent perturbative series

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1.1 Perturbative QED, QCD

Standard perturbative expansions. Asymptotic series

- Before 1952: Expansions in α^n were thought justified, analyticity at $\alpha = 0$ and around seemed to be evident.
- Now: sigularities are admitted (Dyson, followers), expansions in special, singular functions appear more suitable, even desirable. We perform this project.
- Some classes of Feynman diagrams indicate $\sim n!$ at high n. QED: Dyson52, QCD: GrossNev74,Lautr,'tHooft77(79), LeGuillou Zinn-Justin90,Zakharov92,BigiShifm94,Beneke99,!Minkowski!3.1
- *DYSON*: Let a divergent series be asymptotic: Strong ambiguity, 2.5, 3. This calls for additional inputs. IC,JF,IV,JPA **42**(2009); Appl.Num.Math. **60**(2010).
- RECALL: the series $\sum_{n=0}^{\infty} F_n z^n$ is called asymptotic to $F(z)$ as $z \to 0$ on the set S, if the functions $R_N(z) = F(z) - \sum_{n=0}^{N} F_n z^n$ satisfy the condition $R_N(z) = o(z^N)$, for $N = 0, 1,..., z \to 0$, $z \in S$.

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1.2 Perturbative QCD

Summary

- NOTE: Let a function be singular at $z = 0$; the series in z^n may be asymptotic. E.g.: expand $\mathrm{e}^{-1/z}$ in z^n, $x > 0$. All power terms vanish. The series is convergent, but not tending to $\mathrm{e}^{-1/z}$.
- In the hadronic decay of τ, npt power corrections (PC) are suppressed (2.8). BraatenNarisonPich,Nucl.Phys.B373(1992)590.
- Dependence of the coupling $\alpha_s(\mu^2)$ on mass parameter μ (RGE): $\mu^2 \frac{da_s(\mu^2)}{d\mu^2} \equiv \beta(a_s) = -a_s^2 \sum_k \beta_k a_s^k, \quad$ where $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$
- β-function in massless QCD is known to four β_j (loops)
- Adler function in massless QCD; expansion coeffs known to α_s^4
- *RENORMALONS*: Let some classes of Feynman diagrams be divergent (but compensations among the classes aren't excluded, (1),2.2,3.1). Field correlators may be singular at $\alpha = 0$.

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Outline

1. Introduction. Perturbative QCD ... 1.1 - 1.2
2. ADLER FUNCTION $\widehat{D}(\alpha_s)$. Expand $\widehat{D}(\alpha_s)$ in powers of α_s ... 2.1 - 2.6
3. Forget about $\widehat{D}$! Watson's lemma. Expand $B(u)$ in powers of u ... 3.1 - 3.4
4. FORGET ABOUT POWERS! New expansions are needed ... 4.1 - 4.7
5. OCM: $\mathcal{H}$ becomes the disk of convergence...5.1 - 5.5
6. Singularity softening ... 6.1 - 6.3
7. NON-POWER perturbation series. Test on Beneke-Jamin models...7.1 - 7.10
8. Determination of $\alpha_s(M_\tau^2)$ ... 8.1 - 8.4

2.1 Hadronic τ decay

ADLER FUNCTION in massless QCD

- Hadronic width of the τ lepton:
  $R_\tau = \frac{\Gamma[\tau\to\nu_\tau, had, (\gamma)]}{\Gamma[\tau\to\nu_\tau, e, \bar{\nu}_e, (\gamma)]} = \frac{\Gamma_{had}}{\Gamma_e} = 3.4771 \pm 0.0084$; (γ) denoting possible additional photons or lepton pairs ALEPH,HeaFlavAverGrp
- INCLUSIVE: Integrating over s, $R_\tau = \frac{1}{\Gamma_e}\int_0^{m_\tau^2} \frac{d\Gamma_{had}}{ds} ds$.
- Optical theorem: express $\sum_{had}\langle 0|J_\mu(q)|had\rangle\langle had|J_\nu^\dagger(q)|0\rangle$, the hadron production matrix element, in the v.e.v.s of the product of two weak currents Im $\langle 0|J_\mu(q)J_\nu^\dagger(q)|0\rangle$. Decomposing the hadron tensor into Lorentz covariants, we get

  $$\boxed{i\int dx\, e^{iqx}\,\langle 0|\, T\{J_\mu(x)\, J_\nu^\dagger(0)\}|0\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi(s)}, \quad J_\mu = \bar{\psi}_u\gamma_\mu\psi_d$$

- Adler function: $$\boxed{\widehat{D}(\alpha_s) = -\, s\,\frac{d}{ds}\left[\Pi(s)\right] - 1}$$
- (we shall write either $\widehat{D}(\alpha_s)$ or $\widehat{D}(s)$).

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2.2 Dominant perturbative QCD correction

τ decay rate. The use of the Cauchy theorem in the s plane to obtain (3)

- In the part of R_τ produced by V or A current, npt corrections are suppressed. BraatenNarisonPich92, Altarelli: *PRECOCIOUS* AF.
  R_τ, the τ decay rate into light quarks, is:

$$R_\tau \; = \; N_c \, S_{\rm EW} \, |V_{ud}|^2 \left[1 + \delta^{(0)} + \delta_{\rm PC} + \delta'_{\rm EW} \right] \qquad (1)$$

- N_c is # of quark colours,
- $S_{\rm EW}$ and $\delta'_{\rm EW}$ are EW corrections,
- $\delta^{(0)}$ is the dominant pt QCD correction, BrNaPi,LeDibPi92.
- $\delta_{\rm PC}$ is quark mass and higher dim. operator (condensate) corrections

$$R_\tau \sim 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) {\rm Im}\, \Pi(s) \qquad (2)$$

$$\delta^{(0)} = \frac{1}{2\pi i} \oint_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) \widehat{D}_{pert}(s) \qquad (3)$$

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2.3 Coefficients of standard expansion

ADLER FUNCTION, see $\widehat{D}(\alpha_s)$

Insert (4) to (3), represented as RG improved series

$$\widehat{D}(s) = \sum_{n=1}^{\infty} K_n (a_s(s))^n. \tag{4}$$

In the $\overline{MS}$ scheme, for $n_f = 3$, the coeffs K_j calculated up to now ($j = 1, 2, 3, 4$) are

$$K_1 = 1, K_2 = 1.6398, K_3 = 6.3712, K_4 = 49.076. \tag{5}$$

$$\widehat{D}_{pert}(\alpha_s) = \sum_{n\geq 1} \left[c_{n,1} + \sum_{k=2}^{n} k\, c_{n,k} \ln^{k-1}(-s/\mu^2) \right] (a_s(\mu^2))^n \ ,$$

where • $c_{j,1} = K_j$, (*Baikov* 2008)

• $c_{n,k}$ for $2 \leq k \leq n$ depend on $c_{n,1}$ and the coefficients β_k,

$$\mu^2 \frac{d a_s(\mu^2)}{d\mu^2} \equiv \beta(a_s) = -a_s^2 \sum_k \beta_k a_s^k, \tag{6}$$

$\beta_0 = 9/4,\ \beta_1 = 4,\ \beta_2 = 10.0599,\ \beta_3 = 47.228$. See 1.2

van Ritbergen, Vermaseren, Larin 1997, Czakon 2005

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2.4 Current choices of summation in pt QCD

Summation methods, FOPT ..... (2.2)

- **CURRENT CHOICES OF SUMMATION:** $\widehat{D}$ depends on s, $\widehat{D}_{pert}$ on s and the renorm.scale μ^2 (through α_s & the pt logs). Den. $a = \alpha_s(\mu^2)/\pi$, $L = \ln(-s/\mu^2)$. Choosing a fixed scale $\mu^2 = M_\tau^2$, we get Fixed Order PT,
- **FOPT:** expand $\widehat{D}_{FOPT}$, truncate and put $\mu^2 = M_\tau^2$ (Ben11). We get $\widehat{D}_{FOPT}(a, L) = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k c_{n,k} L^{k-1}$, the $c_{n,1}$ from Feynman diagrams. !Badly behaved near the cut; we therefore try Contour Improved PT:
- **CIPT:** solve the RGE (6),1.2 ($n_f = 3$, four loops are known), $\mu^2 = -s$ Piv91,LeDibPich92,11. Insert (4) to (3), integrate. The Taylor expansion of $a_s(s)$ is $\sum_{j\geq 1} \xi_j (a_s(s_1))^j$, the ξ_j depending on $\eta_1 = \ln(s/s_1)$ and β_j. CI summation is obtained:

$$\widehat{D}_{CIPT}(\alpha_s(-s)/\pi, 0) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi} \right)^n \tag{7}$$

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2.5 Discussion. Solution is ambiguous

Renormalons

In conclusion, a set of 1st order linear differential equations for the functions $D_n(aL)$, see

CONCLUSION: The expansion of $B(u)$ in u^n has better h.o. properties than that of $\widehat{D}(\alpha_s)$ in the α_s^n, see sec. 3.

$\widehat{D}(s)$ is not unique, being obtained from $B(u)$ by an integral of the Laplace-Borel type, with a great variety of contours. $B(u)$ has singularities along the two real semiaxes of the the u-plane

BrownYaffeZhai92; Zakharov92; IC,JF,IVrkocJPA09,ANM10

- $u \geq 2$ (IR renormalons),
- $u \leq -1$ (UV renormalons), and
- instanton-antiinstanton pairs along the positive real semiaxis.

The integration needs a prescription. We take the PV,

$$\widehat{D}_{\mathrm{CIPT}}(s) = \frac{1}{\beta_0}\mathrm{PV}\int_0^\infty \exp\left(\frac{-u}{\beta_0 a_s(-s)}\right)B(u)\mathrm{d}u. \qquad (8)$$

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Outline

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3.1 AMBIGUITIES. WATSON's LEMMA

1 Large-order behaviours of $\Phi^{\mu,\nu}_{0,c}(\lambda)$ and of $B(u)$ are different. GENERAL case: μ, ν positive

- $\widehat{D}(\alpha_s)$ is singular at $\alpha_s = 0$, because
- 1/ Some classes of Feynman diagrams have ♣ $c_{n,1} \sim n!$
  2/ 't Hooft 1979
  3/ $B(u)$ has renormalons at finite distance from $u = 0$, see 2.8
- $\Rightarrow$ But: 1 Possible compensations among classes? See 1.2, 2.2
  2 Minkowski?? 1.1
- Asymptotic expansions do not determine the function uniquely:
- **WATSON's LEMMA** (λ is interpreted as $1/\alpha_s$)

$$\Phi^{\mu,\nu}_{0,c}(\lambda) = \int_0^c \exp(-\lambda u^\mu) u^{\nu-1} B(u) du \tag{9}$$

with $c, \mu, \nu > 0$. Let $B(u) \in C^\infty[0, c]$ and $B^{(k)}(0)$ be $\lim_{u\to 0_+} B^{(k)}(u)$. Let ε satisfy $0 < \varepsilon < \pi/2$. Then for $\lambda \to \infty$, $\lambda \in S_\varepsilon = |\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$, the asymptotic expansion holds:

$$\Phi^{\mu,\nu}_{0,c}(\lambda) \sim \frac{1}{\mu} \sum_{n=0}^{\infty} \lambda^{-(n+\nu)/\mu} \Gamma((n+\nu)/\mu) \frac{B^{(n)}(0)}{n!}. \tag{10}$$

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3.2 What follows from Watson's lemma

2 Large-order behaviours of $\widehat{D}_c(\alpha_s)$ and of $B(u)$ are different. SPECIAL case $\mu = \nu = 1$

- NOTE: The r.h.s. of (10) is c independent and the integration contour from 0 to c is arbitrary, see Caprini, JF, Vrkoč 09, 10
  $\Rightarrow$ INFINITE AMBIGUITY OF PT $\Leftarrow$
- For $\mu = \nu = 1$ and $c \to \infty$, the $\Phi_{0,c}$ are special cases of (9, 10):

$$\widehat{D}_c(\alpha_s) = \int_0^c \exp(-u/\alpha_s) B(u) du \qquad (11)$$

  Take $B(u)$ holomorphic, except the rays $u \leq -1$ and $u \geq 2$, see 2.10. If $B(u) = \sum_0^\infty u^n B^{(n)}(0)/n!$, then (11) (principal value) and

$$\widehat{D}_c(\alpha_s) \sim \sum_{n=0}^{\infty} \alpha_s^{n+1} \Gamma(n+1) \frac{B^{(n)}(0)}{n!} \qquad (12)$$

- **NOTE $\Gamma(n+1)$ in (12):** The expansion coeffs for $B(u)$ behave **by $\frac{1}{n!}$ tamer** than those in (12) for $\widehat{D}_c(\alpha_s)$. Convergence radius is $\neq 0$, the point $u = 0$ lies *INSIDE* convergence circle. See 3.1, 3.3, 5.3.

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3.3 Power expansions: Choose $B(u)$ rather than $\widehat{D}(\alpha_s)$

Watson's lemma, Discussion

(12): The coeffs for $\widehat{D}(\alpha_s)$ behave $\sim n!$ at high n. Those for $B(u)$ have a better behaviour. See 3.1, 3.2.

- 1 Example: Set $B(u) = u^n$. The r.h.s. of (11) becomes $n!\alpha_s^{n+1}$:

  ♣ **u^n in the u plane generates $n!\alpha_s^{n+1}$ in the α_s plane** ♣

- 2 *RECALL* (2.9): QED is analogous (Dyson 1952), but NUMERICAL precision is not affected, α being small at the usual scales.
- 3 QCD: $\alpha_s(M_\tau^2)$ is big, so these facts DO MATTER in practical predictions
- 4 *WE KNOW*: It is better to deal with $B(u)$, which is easier to handle.
- 5 Thus, to reach 2.5(blue), we arrive at the ♣**CONCLUSION:**
- $B(u)$ is preferable to $\widehat{D}(\alpha_s)$. To expand $B(u)$, *FIND* AN OPTIMAL SET of expansion functions yielding
  - The fastest convergence rate at high n and
  - A high (numerical) accuracy at low n.
- Let me disclose: The function set are *NOT* the POWERS u^n; see below

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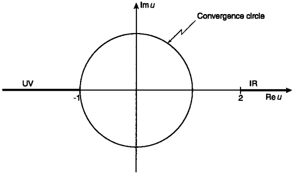
3.4 Special case: Adler function $\widehat{D}(s)$

$B(u)$ encodes the high-order increase of coeffs by singularities in the u-plane. See 2.8, 6.5

Take $B(u)$ expanded in u^n :

$$B(u) = \sum_{n=0}^{\infty} b_n u^n, \quad b_n = \frac{c_{n+1}}{\beta_0^n\, n!} \quad \Rightarrow \quad \widehat{D}(s) = \frac{1}{\beta_0} \int_0^\infty e^{-u/(\beta_0 a_s(s))}\, B(u)\, du$$



- $B(u)$ has singularities along $u \leq -1$ and $u \geq 2$ (UV and IR renormalons)
  - Laplace-Borel integral is not defined, a prescription is needed:

$$\widehat{D}(s) = \frac{1}{\beta_0}\, PV \int_0^\infty e^{-u/(\beta_0 a_s(s))}\, B(u)\, du$$

.

- Exact results: $B(u) \sim (1+u)^{-\gamma_1}$ and $B(u) \sim (1-u/2)^{-\gamma_2}$, where γ_1, γ_2 are known; see 7.1, 8.1 Mueller 1985, Beneke, Braun, Kivel 1997

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4.1 Adjusting the expansion functions to the singularities

Standard power series and the non-power ones, see eg 8.7

♠ Standard **POWER SERIES** *Beneke Jamin* [BJ, 2008]:

- ♣EXPAND $\widehat{D}(\alpha_s)$ in POWERS of α_s, CONFRONT FOPT with CIPT.
- **RESULT** [BJ]: ♣ **FOPT is superior to CIPT**

♠ **NONPOWER SERIES** [CaFi, 2009]:

- REPLACE the α_s^n by the $\mathcal{W}_n(\alpha_s)$ having similar singularities as $\widehat{D}(\alpha_s)$.
- NON-POWER Perturbation Theory (NPPT) consists of:
  - optimal conf. mapping (OCM, CiFi, CaFi, 3.4) of the Borel u-plane, and
  - singularity softening (SS, SoSu) in terms of the $\mathcal{W}_n(\alpha_s)$ functions.
- ♣EXPAND $\widehat{D}(\alpha_s)$ in the $\mathcal{W}_n(\alpha_s)$, CONFRONT FOPT with CIPT.
- **RESULT** [CaFi]: ♣ **CIPT is superior to FOPT** (opp. to [BJ], see 8.7).

- *WHAT ABOUT* $B(u)$**?**

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4.2 A NEW EXPANSION of $\widehat{D}(\alpha_s)$ is needed

1 Symmetria symmetriam invocat

- **The singularities of $\widehat{D}(\alpha_s)$, if encoded in renormalons at FINITE distance from $u = 0$, are located AT $\alpha_s = 0$.**
- **A serious defect of Standard PT:** The singular $\widehat{D}(\alpha_s)$ is expanded in the α_s^n, which, analytic in α_s, *DO NOT SEE* the physics hidden in the singularities of $\widehat{D}$. So, some ESSENTIAL PHYSICS may be INVISIBLE to standard PT. (L'essentiel est invisible pour la theorie des perturbations.) Each approximant is analytic, in contrast to $\widehat{D}(\alpha_s)$.
- THE FORM of our expansion functions depends on which property of $\widehat{D}$, a physical quantity, is considered important: symmetry, analyticity, singularities, etc. We therefore have to construct expansion functions having the same property:

  ♣ **Every construction calls for specific building constituents**,
  ♣ **Every function calls for specific expansion functions**
  See also 9.1 - 9.5.

4.3 A NEW EXPANSION of $\widehat{D}(\alpha_s)$ is needed

- Apply this to the *SINGULAR* $\widehat{D}(\alpha_s)$. Expand $\widehat{D}(\alpha_s)$ in a set of functions whose singularities resemble those of $\widehat{D}(\alpha_s)$ as much as possible:
- **Expansion of a SINGULAR function** $\widehat{D}(\alpha_s)$ calls for **SINGULAR expansion functions**, $\mathcal{W}_n(\alpha_s)$ (*).
- NOTE: The Standard PT does NOT observe the demand (*).
  Indeed, $\widehat{D}(\alpha_s)$ is **SINGULAR**,
  while the expansion functions are α_s^n, i.e., **ANALYTIC** .
- This is why the powers α_s^n are not suited for expanding the singular $\widehat{D}(\alpha_s)$. We therefore abandon the powers α_s^n, and build up
  Non-Power Perturbation Theory (NPPT),
  in which the rule (*) is observed (see 8.1, 10.4).

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4.4 APPLY (*) to $B(u)$, rather than to $\widehat{D}(\alpha_s)$

- INSTEAD of expanding ♠ *ADLER FUNCTION* $\widehat{D}(\alpha_s)$ in α_s^n,
  we shall expand **its ♠ BOREL TRANSFORM** $B(u)$ **in** u^n
- *WHY* do we do so? *WHAT IS BETTER*, and why? The expansion of $\widehat{D}$ and B behaves like $n!\alpha_s^{n+1}$ and u^n respectively. In more detail:
- *REMIND* Watson's lemma. You have (page 3.2) two series:

$$B(u) = \textstyle\sum_0^\infty u^k B^{(k)}(0)/k!$$

$$\widehat{D}(\alpha_s) \sim \sum_{k=0}^{\infty} \alpha_s^{k+1} \Gamma(k+1) \frac{B^{(k)}(0)}{k!}.$$

Note the factor $\Gamma(k+1)$. We use the formula

$$\mathbf{1}/\rho_{\mathbf{B}} = \limsup_{\mathbf{n}\to\infty} \sqrt[n]{|a_n|},$$

for $1/\rho_B$, and $1/\rho_D = \limsup_{n\to\infty} \sqrt[n]{n!|a_n|}$ for $1/\rho_D$, where ρ_B and ρ_D is the convergence radius of the two series respectively. If ρ_B is finite nonvanishing ($= 1$ for the UV renormalon), ρ_D is **zero, and the expansion of** $\widehat{D}(\alpha_s)$ **diverges for every** $\alpha_s \neq 0$.

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4.5 APPLY (*) to $B(u)$, rather than to $\widehat{D}(\alpha_s)$

4 Expansion of a *SINGULAR* function $B(u)$ calls for *SINGULAR* expansion functions, $(\tilde{w}(u))^n$

- Unlike in the α_s-plane, the boundary of $\mathcal{H}$ in the u-plane NEVER TOUCHES the orign. The expansion of $B(u)$ in u^n has much a milder h.o. behaviour than that of $n!\alpha_s^n$(factor $n!$). There are two steps of improvement:
- STEP 1: Do NOT EXPAND $\widehat{D}(\alpha_s)$ in α_s^n,
  but rather **EXPAND** $B(u)$ in u^n. *HOWEVER*,
- STEP 2 is an improvement of Step 1: EXPAND $B(u)$ **NOT in** u^n**,** *but in the* $(\tilde{w}(u))^n$ **(OCM, optimal conformal mapping),** $\tilde{w}(u)$ **having the same location of singularities as** $B(u)$.
- *CONCLUSION*: Rather than in u^n, expand $B(u)$ in the $(\tilde{w}(u))^n$, since $\tilde{w}(u)$ maps $\mathcal{H}$ onto the DISK $|\tilde{w}(u)| \leq 1$. *INTUITIVELY*: As $\tilde{w}(u)$ and $B(u)$ have identical location of singularities, the expansion of B in the $\tilde{w}^n$ is simpler and easier to handle than that in the u^n. EXACTLY sec. 6. To make full use of analyticity,

USE OCM in the Borel plane.

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4.6 OCM makes $\mathcal{H}$ COINCIDE with the convergence disk

see 4.8, 6.5 and the sections 5 and 6

- ♣ Let $\mathcal{H}$ be the analyticity domain of $B(u)$ in the u-plane,
  ♣ Let $\mathcal{G}_w$ be a subset (region) of $\mathcal{H}$: $\mathcal{G}_w \subset \mathcal{H}$,
  ♣ Let $w(u)$ map $\mathcal{G}_w$ conformally onto $|w(u)| < 1$
  ♣ Let $\tilde{w}(u)$ map $\mathcal{G}_{\tilde{w}} = \mathcal{H}$ conformally onto $|\tilde{w}(u)| < 1$
- In QCD, the singularities of $B(u)$ are located along the rays $u > 2$ and $u < -1$ (IR and UV renormalons). $\mathcal{H}$ is the u-plane cut along the rays; see 4.10 and 6.5 for details
- ***IF the power series of a function analytic in u is DIVERGENT at some $u \in \mathcal{H}$, convergence is **restored by OCM at that point*****.
- ***IF the power series is CONVERGENT at some $u \in \mathcal{H}$, the convergence rate is **enhanced by OCM at that point*****.
- This is impossible in the α_s-plane, where the point $\alpha_s = 0$ *MAY APPEAR ON THE BOUNDARY* of the analyticity domain. See 4.8. This is why **$B(u)$ should be preferred to $\widehat{D}(\alpha_s)$, and Expansions of $B(u)$ in $(\tilde{w}(u))^n$ should be preferred to those in u^n**.

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4.7 Discussion

See also 2.10, 4.5, 4.6, 6.5 and sec. 5 for details

- Take $\mathcal{G}_w \subset \mathcal{H}$. If $\mathcal{G}_w = \mathcal{H}$, we write $\tilde{w}(u)$, $\mathcal{G}_{\tilde{w}}$. The equality $\mathcal{G}_{\tilde{w}} = \mathcal{H}$ implies that the whole doubly-cut u-plane (which is simply-connected) is mapped onto the $|\tilde{w}(u)| < 1$ disk, $\tilde{w}(u)$ having the same *LOCATION* of singularities as $B(u)$. The disk $|\tilde{w}(u)| < 1$ is the

  DISK OF CONVERGENCE of the expansion of $B(u)$ in the $(\tilde{w}(u))^n$,
  and IS IDENTICAL
  with $\mathcal{H}$, the REGION OF ANALYTICITY of $B(u) = B(u(\tilde{w}))$.

- See **Lemma 1** and **Lemma 2**, sec. 6

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4.8 Power expansions and conformal mappings

Optimal mapping, optimal expansion

- 1 Let $\widehat{D}(\alpha_s)$ (analytic, but with singularities) be expanded in powers of α_s. Assume that $\alpha_s = 0$ lies on (or near) the boundary of the analyticity domain. How to get along the bad convergence properties?
- 2 The expansion coeffs of $\widehat{D}(\alpha_s)$ grow fast, $\Gamma(k+1)\frac{B^{(k)}(0)}{k!} = B^{(k)}(0)$ at high orders, while those of $B(u)$ behave $\frac{1}{k!}$ **times tamer**, $\frac{B^{(k)}(0)}{k!}$.
- 3 $B(u)$ is holomorphic in $\mathcal{H}$, in the u-plane cut along $u > 2$ (IR) and $u < -1$ (UV). Note that
  ♣$\tilde{w}(u)$ CONFORMALLY MAPS $\mathcal{H}$ ONTO the disk $|\tilde{w}(u)| < 1$ ♣
- 4 Expand $B(u)$ in $(\tilde{w}(u))^n$, NOT in u^n. This expansion
  is CONVERGENT in the whole $\mathcal{G}_{\tilde{w}} = \mathcal{H}$, and
  has the FASTEST *CONVERGENCE RATE* at all points of $\mathcal{H}$.
- 5 The mapping $\tilde{w}(u)$ and the expansion of $B(u)$ in powers of $\tilde{w}(u)$ are called **OPTIMAL**; Ciulli & JF, Nucl.Phys.1961 See below sec. 7.

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4.9 Merits and virtues of (OCM)

What does the method of optimal conformal mapping (OCM) of the region of analyticity do?

- OCM enlarges the region of convergence, and
- In the original region of convergence, OCM enhances the convergence rate of the original expansion
- **INTUITIVE EXPECTATION:**
  ♠ *THE LARGER $\mathcal{G}_w$, THE FASTER THE CONVERGENCE RATE* ♠
  PROOF: Ciulli,Nucl.Phys.**24**,465(1961), CaFi, Phys.Rev.D**84**(2011)
- Take $w(u)$ such that $\mathcal{G}_w = \mathcal{G}_{\tilde{w}} = \mathcal{H}$ (OCM), $\mathcal{G}_w$ not exceeding $\mathcal{H}$; otherwise a singularity is pressed inside the circle, and convergence deteriorates.
- **OCM maps $\mathcal{H}$ onto the disk $|\tilde{w}(u)| < 1$**. OCM is important, the expansion of $B(u)$ in the $(\tilde{w}(u))^n$ has the fastest convergence rate.
- **WE CONCLUDE: any singularity of $B(u)$ located at finite nonvanishing distance** from $u = 0$ **produces a singularity of $\widehat{D}(\alpha_s)$ at $\alpha_s = 0$. This happens if, e.g., $B(u)$ has renormalons**. If $\widehat{D}(\alpha_s)$ has no singularity at $\alpha_s = 0$, all renormalons are at infinite distance from $u = 0$.

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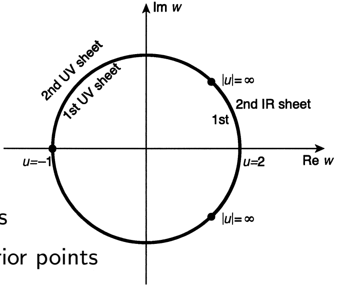
Outline

5.1 Optimal conformal mapping in the u-plane

- Map the cut u-plane onto the disk $|w| < 1$ in the plane $w = \tilde{w}(u)$, with $\tilde{w}(0) = 0$ (see also 3.4,10.4)

$$\tilde{w}(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}$$



- The series $B(u) = \sum c_n\, w^n, \quad w = \tilde{w}(u)$
- converges in the whole u-plane up to the cuts
- best asymptotic convergence rate for all interior points
- makes full use of our knowledge of the location of the singularities of $B(u)$
- suggests the expansion: $\widehat{D}(\alpha_s) = \sum\limits_n c_n \frac{1}{\beta_0} PV \int\limits_0^\infty e^{-u/(\beta_0 a_s(s))} (\tilde{w}(u))^n du$

What are the advantages of OCM in applications?

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5.2 Lemmas 1 and 2

Ciulli 1961, Caprini, Fischer 2011. See 3.4 and 3.6 for motivation

Lemma 1: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two domains in the complex u-plane, with $\mathcal{D}_2 \subset \mathcal{D}_1$, $\mathcal{D}_2 \neq \mathcal{D}_1$. Consider the two conformal mappings

$$z_1 = \tilde{z}_1(u) : \mathcal{D}_1 \to \mathcal{K}_1 = \{z_1 : |z_1| < 1\}, \;\; z_2 = \tilde{z}_2(u) : \mathcal{D}_2 \to \mathcal{K}_2 = \{z_2 : |z_2| < 1\}$$

Let Q be a point of $\mathcal{D}_2$, $Q \in \mathcal{D}_2$, such that $\tilde{z}_1(Q) = 0$ and $\tilde{z}_2(Q) = 0$. Then

$$|\tilde{z}_1(u)| < |\tilde{z}_2(u)|, \quad \text{for all} \;\; u \in \mathcal{D}_2, \;\; u \neq Q. \tag{13}$$

Lemma 2: Let $B(u)$ be a function holomorphic in $\mathcal{D}_1$ and the expansions

$$B(u) = \sum_{n=0}^{\infty} b_{n,1}(\tilde{z}_1(u))^n, \quad B(u) = \sum_{n=0}^{\infty} b_{n,2}(\tilde{z}_2(u))^n, \tag{14}$$

be convergent for $z_1 \equiv \tilde{z}_1(u) \in \mathcal{K}_1$ and $z_2 \equiv \tilde{z}_2(u) \in \mathcal{K}_2$ respectively. Assume also that

$$\lim_{n\to\infty} \sqrt[n]{|b_{n,1}|} = \lim_{n\to\infty} \sqrt[n]{|b_{n,2}|} = 1. \tag{15}$$

Then a positive integer $N = N(u)$ exists such that the following inequality

$$\mathcal{R}_n(u) = \left| \frac{b_{n,1}(\tilde{z}_1(u))^n}{b_{n,2}(\tilde{z}_2(u))^n} \right| < 1, \tag{16}$$

holds for any n integer, $n > N$, and $u \in \mathcal{D}_2$, $u \neq Q$.

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5.3 Proof of Lemma 1

- Define $F(z_2) = \tilde{z}_1(\tilde{z}_2^{[-1]}(z_2))$ for $z_2 \in \mathcal{K}_2$, where $\tilde{z}_2^{[-1]}$ is the inverse to $\tilde{z}_2$
  - $F(z_2)$ is holomorphic on the unit disk $\mathcal{K}_2$ of the z_2-plane and maps this disk into the unit disk $\mathcal{K}_1$ of the z_1-plane, *i.e.* $|F(z_2)| \leq 1$
  - Since $\tilde{z}_1(Q) = 0$ and $\tilde{z}_2(Q) = 0$, it follows that $F(0) = 0$
- Schwarz's lemma[1] gives $|F(z_2)| \leq |z_2|$ for $z_2 \in \mathcal{K}_2$
- The definition of $F(z_2)$ and the obvious relation $\tilde{z}_2^{[-1]}(z_2) = u$ for $u \in \mathcal{D}_2$ leads to $|\tilde{z}_1(u)| \leq |\tilde{z}_2(u)|, \quad u \in \mathcal{D}_2$
- Ignoring mappings that reduce to mere rotations, we obtain (13), which proves Lemma 1.

[1] **Schwarz's lemma:** If a function $F(z)$ is holomorphic on the disk $|z| < 1$ and satisfies the conditions $F(0) = 0$ and $|F(z)| < 1$ for $|z| < 1$, then $|F(z)| \leq |z|$ everywhere in $|z| < 1$. If the equality sign occurs at least at one interior point, then it takes place everywhere and $F(z)$ has the form $F(z) = z\exp(i\alpha)$ with α real.

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5.4 Proof of Lemma 2

- The relations (15) imply that the coefficients $|b_{n,j}|$ can, for large enough n, be represented in the form

$$|b_{n,j}| = e^{g_j(n)}, \qquad j = 1, 2, \tag{17}$$

  where $g_j(n)$ are real-valued functions, with $\lim_{n\to\infty} g_j(n)/n = 0$, $j = 1, 2$.

- The ratio defined in (16) can be written as

$$\mathcal{R}_n(u) = e^{g(n)} \times (\rho(u))^n, \tag{18}$$

  where

$$g(n) = g_1(n) - g_2(n), \qquad \rho(u) = |\tilde{z}_1(u)/\tilde{z}_2(u)|. \tag{19}$$

- Taking the logarithm of (18), one obtains

$$\ln \mathcal{R}_n(u) = n \left[\frac{g(n)}{n} + \ln \rho(u) \right], \tag{20}$$

- From (19) it follows that $\lim_{n\to\infty} g(n)/n = 0$, while $\rho(u) < 1$ for all $u \in \mathcal{D}_2$, $u \neq Q$, according to Lemma 1.
- This implies that, at large n, $\ln \mathcal{R}_n(u) < 0$, proving Lemma 2.

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Outline

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6.1 Singularity Softening

see also 6.2

- The series $B(u) = \sum c_n (w(u))^n$ accounts for the LOCATION of the singularities of $B(u)$; $w(u)$ is optimal for $w = \tilde{w}$, where $\mathcal{G}_{\tilde{w}} = \mathcal{H}$:
  $\tilde{w}(u) = (\sqrt{1+u} - \sqrt{1-u/2})/(\sqrt{1+u} + \sqrt{1-u/2})$
- Something is known also about the *NATURE* of the singularities of $B(u)$ (5.6,9.1). Softening: the two known singularities in

$$B(u) = \frac{1}{(u+1)^{\gamma_1}(2-u)^{\gamma_2}} \sum_{n=0}^{N} c_n \tilde{w}^n \qquad (21)$$

  can be suppressed by suitable factors; the two singularities become weaker or disappear. This procedure is not exact, unlike OCM. Its aim is to make the singularities softer, less influential; it can be extended for the next known singularities in the u-plane in powers of

$$\tilde{w}_{jk}(u) = \frac{\sqrt{1+u/j} - \sqrt{1-u/k}}{\sqrt{1+u/j} + \sqrt{1-u/k}}, \qquad j \geq 1, \quad k \geq 2$$

- $\tilde{w}_{jk}$ maps the u-plane cut for $u < -j$ and $u > k$ onto $|w_{jk}| < 1$

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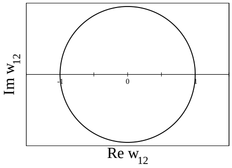
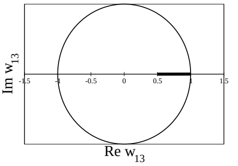
6.2 Various conformal mappings

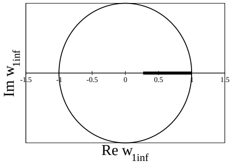
see also 6.1



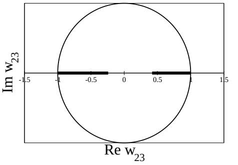
$\tilde{w}_{12}(u)$ (IC&JF1999)



$\tilde{w}_{13}(u)$ (Cvetic, Lee 2001)



$\tilde{w}_{1\infty}(u)$ (Mueller 1992)



$\tilde{w}_{23}(u)$

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6.3 A remark on Proof of Lemma 1

Remark:. Let $\sum_{n=0}^{N} a_n z^n$ be the sum of a (truncated) series in z^n and let $\rho = |z_0|$ be its radius of convergence. Then, for any $|z| < \rho$ (i.e., inside the convergence circle), there is a number q satisfying the inequalities $0 \leq q \leq 1$ such that $|z| \leq q\rho$ (note that q depends on $|z|$).

Inside the convergence circle, the general term $a_n z^n$ tends to zero with increasing order n. A number $A > 0$ exists such that $|a_n z_0^n| = |a_n|\rho^n < A$ for any n. The absolute values $|a_n z^n|$ of all terms are less than the corresponding terms of a decreasing geometrical series of positive numbers. Thus, for any n positive integer, the inequality

$$|a_n z^n| = |a_n z_0^n||z/z_0|^n \leq A q^n,$$

holds, where $q = |z/z_0| = |z|/\rho$ is the quotient of the geometrical progression, $0 \leq q \leq 1$.

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Outline

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7.1 Non-power perturbative expansion functions 1

Notation

- FO version: $$\widehat{D}(s) = \sum_n \tilde{c}_n^{(jk)}(s)\,\tilde{\mathcal{W}}_n^{jk}(\mu^2)$$

$$\tilde{\mathcal{W}}_n^{jk}(\mu^2) = \frac{1}{\beta_0} PV \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(\mu^2))}\,\frac{(\tilde{w}_{jk}(u))^n}{S_{jk}(u)}\,du$$

- CI version: $$\widehat{D}(s) = \sum_n c_n^{(jk)}\,\mathcal{W}_n^{jk}(s)$$

$$\mathcal{W}_n^{jk}(s) = \frac{1}{\beta_0} PV \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(s))}\,\frac{(\tilde{w}_{jk}(u))^n}{S_{jk}(u)}\,du$$

- $S_{jk}(u) = \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(-1)}\right)^{\gamma_1'} \left(1 - \frac{\tilde{w}_{jk}(u)}{\tilde{w}_{jk}(2)}\right)^{\gamma_2'}$
    - $\gamma_1' = \{2\gamma_1 \ \mathit{for}\ j = 1; \quad \gamma_1 \ \mathit{for}\ j \neq 1\}$
    - $\gamma_2' = \{2\gamma_2 \ \mathit{for}\ k = 2; \quad \gamma_2 \ \mathit{for}\ k \neq 2\}$ see 7.1, 8.1

7.2 Non-power perturbation expansion functions 2

- The expansion functions $\mathcal{W}_n^{jk}$ are singular at $\alpha_s = 0$. Their expansions in the α_s^n are divergent
- For $\alpha_s \to 0_+$, the $\mathcal{W}_n^{jk}(\alpha_s) \sim \alpha_s^n$.
- The optimal expansion $\widehat{D}(s) = \sum_n c_n^{12} \mathcal{W}_n^{12}(s)$ is, under certain conditions, convergent in a domain of the complex s-plane (method of steepest descent, IC& JF 2001)
- This however does not imply that this is the correct sum of the series. Convergence is a necessary, not a sufficient condition for finding the correct solution. See also FOPT and CIPT 2.5 - 2.8.

x

7.3 Non-power expansions in Beneke-Jamin models

Beneke, Jamin 2008, Jamin 2011, IC, JF 2009, 2011

1 To test our method and compare it with the standard approach, we need models with a large enough number of terms. We take the models by Beneke and Jamin:

- $\widehat{D}(a_s) = \frac{1}{\beta_0} PV \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(s))} B(u)\,\mathrm{d}u$
- $B(u) = B_1^{\mathrm{UV}}(u) + B_2^{\mathrm{IR}}(u) + B_3^{\mathrm{IR}}(u) + \cdots d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u + \cdots$

$$B_p^{\mathrm{IR}}(u) = \frac{d_p^{\mathrm{IR}}}{(p-u)^{1+\tilde{\gamma}}}\left[1 + \tilde{b}_1(p-u) + \tilde{b}_2(p-u)^2 + \ldots\right]$$

$$B_p^{\mathrm{UV}}(u) = \frac{d_p^{\mathrm{UV}}}{(p+u)^{1+\bar{\gamma}}}\left[1 + \bar{b}_1(p+u) + \bar{b}_2(p+u)^2\right]$$

- The free parameters ($d_1^{\mathrm{UV}}, d_2^{\mathrm{IR}}, d_2^{\mathrm{IR}}, d_0^{\mathrm{PO}}, d_1^{\mathrm{PO}}$, ...) are fixed by reproducing the known values of $c_{n,1}$
  - $d_1^{\mathrm{UV}} = 1.56 \times 10^{-2}$, $d_2^{\mathrm{IR}} = 3.13, \ldots$ Beneke & Jamin, 2008
- Other possibilities were also examined numerically
  - models with larger d_1^{UV} and smaller d_2^{IR} Jamin, IC&JF 2011

x

7.4 The standard and the non-power expansions

in the Beneke-Jamin [BJ] model

- In the BJ model, two lowest singularities of $B(u)$ are assumed, the others being parametrized by polynomials. To confront the two approaches, we use the same model to test our approach (with the $\mathcal{W}_n^{jk}$). A big difference of the two approaches is obtained: We obtain a very good approximation of $\delta^{(0)}$, much faster in the CIPT summation than in the FOPT one.
- In the standard approach, FOPT is preferred to CIPT, but the accuracy is worse. Violent oscillations set on above the 10th-15th order.
- A clean-cut difference between our result and that of [BJ] is obtained (7.5 -7.10): In our (non-power) expansion the exact value of $\delta^{(0)}$ is reached much faster in CIPT than in FOPT. In the standard approach the exact value is reached by FOPT faster than by CIPT; see Refs. [1,4].

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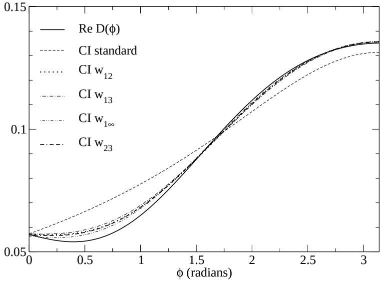
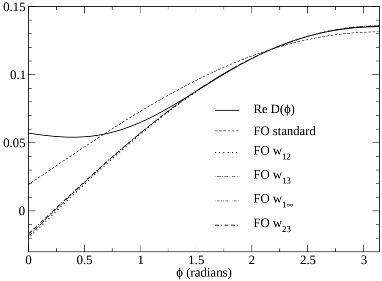
7.5 Confronting non-power expansions with standard ones

1 Real part of $\widehat{D}(s)$ of the model of Beneke and Jamin, $N = 5$



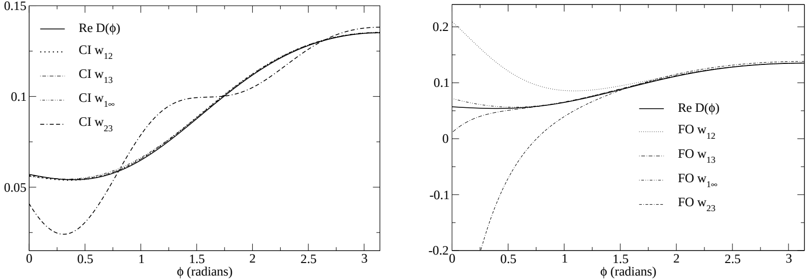


1 The real part of the Adler function of the model of Beneke & Jamin, calculated along the circle $s = M_\tau^2 \exp(i\phi)$ for $\alpha_s(M_\tau^2) = 0.3156$, using the perturbative expansions with $N = 5$ terms.

Left panel: CI expansions. Right panel: FO expansions.

7.6 Confronting non-power expansions with standard ones

2 Real part of $\widehat{D}(s)$ of the model of Beneke and Jamin, $N = 18$



2 As in the previous figure for $N = 18$. The standard CI and FO expansions exhibit big oscillations and are not shown.

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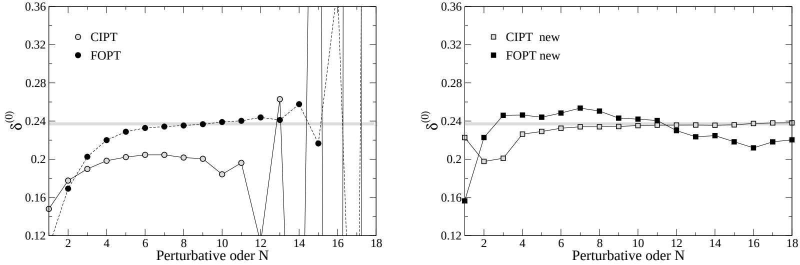
7.7 Confronting non-power expansions with standard ones

3 Integral $\delta^{(0)}$ in the Beneke-Jamin model calculated with standard expansion (left) and with a non-power one (right)



3 Integral $\delta^{(0)}$ calculated for $\alpha_s(M_\tau^2) = 0.34$ with the standard (left, see Beneke and Jamin, IHEP, 2008) and the non-power (right, see Caprini and Fischer, EPJ C, 2009, [4]) CI and FO expansions.

x

7.8 Confronting non-power expansions with standard ones

4 Discussion of Fig. 8.7

EXPANSION	PREFERS the summation	TO	Remark
:			
Standard, in α_s^n	FOPT	CIPT	striking difference
:			
Non-power, in $\mathcal{W}_n^{j,k}(\alpha_s)$	CIPT	FOPT	

- While the standard pt expansion in powers of α_s prefers FOPT to CIPT, expansions in functions of the $\mathcal{W}_n$ type clearly prefer CIPT. This implies that the singularities of $\widehat{D}$ in α_s do play an important role and should not be ignored, as usually happens when the standard series in the α_s^n are used. It is surprising how much the convergence properties are improved when renormalon sigularities are taken into account.
- An important fact is that the non-power approach eliminates oscillations, which grow dramatically with growing order of the α_s^n expansion.

7.9 Confronting non-power expansions with standard ones

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N	CI st.	FO st.	CI w_{12}	FO w_{12}	CI w_{13}	FO w_{13}	CI $w_{1\infty}$	FO $w_{1\infty}$	CI w_{23}	FO w_{23}
2	0.1776	0.1692	0.1977	0.2228	0.2070	0.2203	0.1883	0.2524	0.2123	0.2099
3	0.1898	0.2026	0.2009	0.2460	0.2030	0.2440	0.1975	0.2530	0.2028	0.2437
4	0.1983	0.2200	0.2263	0.2463	0.2194	0.2460	0.2288	0.2465	0.2206	0.2463
5	0.2022	0.2288	0.2290	0.2440	0.2268	0.2423	0.2310	0.2427	0.2292	0.2423
6	0.2046	0.2328	0.2324	0.2484	0.2306	0.2421	0.2321	0.2431	0.2319	0.2449
7	0.2046	0.2342	0.2339	0.2536	0.2331	0.2457	0.2333	0.2454	0.2345	0.2502
8	0.2017	0.2353	0.2339	0.2505	0.2343	0.2484	0.2341	0.2471	0.2347	0.2476
9	0.2004	0.2367	0.2341	0.2431	0.2348	0.2457	0.2346	0.2465	0.2347	0.2377
10	0.1842	0.2390	0.2351	0.2420	0.2348	0.2394	0.2348	0.2436	0.2353	0.2337
11	0.1962	0.2402	0.2359	0.2406	0.2348	0.2352	0.2349	0.2399	0.2348	0.2335
12	0.1123	0.2436	0.2362	0.2298	0.2351	0.2349	0.2349	0.2370	0.2374	0.2262
13	0.2629	0.2408	0.2362	0.2229	0.2355	0.2341	0.2349	0.2356	0.2348	0.2226
14	-0.2915	0.2575	0.2364	0.2242	0.2361	0.2303	0.2349	0.2354	0.2395	0.2314
15	1.1011	0.2170	0.2367	0.2173	0.2366	0.2277	0.2350	0.2357	0.2356	0.2365
16	-3.362	0.3818	0.2368	0.2102	0.2369	0.2305	0.2351	0.2360	0.2343	0.2374
17	9.5931	-0.1881	0.2368	0.2176	0.2372	0.2356	0.2352	0.2360	0.2533	0.2512
18	-31.52	2.144	0.2368	0.2201	0.2373	0.2371	0.2354	0.2359	0.1926	0.2665

5 The quantity $\delta^{(0)}$ for the model proposed by Beneke & Jamin 2008 calculated for $\alpha_s(M_\tau^2) = 0.34$ with the standard and modified CI and FO expansions truncated at the order N. **Exact value $\delta^{(0)} = 0.2371$**

x

7.10 Confronting non-power expansions with standard ones

6

N	CI st.	FO st.	CI w_{12}	FO w_{12}	CI w_{13}	FO w_{13}	CI $w_{1\infty}$	FO $w_{1\infty}$	CI w_{23}	FO w_{23}
2	0.1776	0.1692	0.1977	0.2228	0.2070	0.2203	0.1883	0.2524	0.2123	0.2099
3	0.1898	0.2026	0.2009	0.2460	0.2030	0.2440	0.1975	0.2530	0.2028	0.2437
4	0.1983	0.2200	0.2263	0.2463	0.2194	0.2460	0.2288	0.2465	0.2206	0.2463
5	0.2022	0.2288	0.2290	0.2440	0.2268	0.2423	0.2310	0.2427	0.2292	0.2423
6	0.2041	0.2318	0.2263	0.2493	0.2271	0.2420	0.2284	0.2431	0.2260	0.2454
7	0.2041	0.2290	0.2201	0.2628	0.2220	0.2481	0.2230	0.2472	0.2174	0.2580
8	0.2023	0.2213	0.2202	0.2756	0.2164	0.2595	0.2182	0.2541	0.2136	0.2734
9	0.2037	0.2110	0.2175	0.2742	0.2143	0.2686	0.2154	0.2608	0.2138	0.2706
10	0.1924	0.2032	0.2055	0.2709	0.2144	0.2651	0.2146	0.2629	0.2115	0.2517
11	0.2124	0.2004	0.1982	0.2905	0.2136	0.2504	0.2146	0.2578	0.2068	0.2531
12	0.1412	0.2071	0.2007	0.3063	0.2111	0.2406	0.2148	0.2468	0.2081	0.2627
13	0.3121	0.2117	0.2022	0.2820	0.2086	0.2449	0.2149	0.2340	0.2060	0.2133
14	-0.2105	0.2344	0.2001	0.2666	0.2074	0.2459	0.2146	0.2239	0.2124	0.1338
15	1.2336	0.1934	0.2009	0.2865	0.2079	0.2176	0.2142	0.2187	0.2087	0.1192
16	-3.147	0.3500	0.2044	0.2562	0.2091	0.1676	0.2136	0.2175	0.2073	0.0930
17	9.948	-0.2333	0.2059	0.1822	0.2102	0.1355	0.2130	0.2175	0.2275	-0.0415
18	-30.94	2.084	0.2058	0.1722	0.2107	0.1345	0.2124	0.2159	0.1617	-0.1019

6 The quantity $\delta^{(0)}$ for an alternative model with a smaller residue of the first IR renormalon, calculated for $\alpha_s(M_\tau^2) = 0.34$ with the standard and modified CI and FO expansions truncated at the order N. **Exact value $\delta^{(0)} = 0.2102$**

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Outline

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8.1 Non-power perturbative expansions

Conclusions from the numerical tests

- Non-power pt expansions provide :
- Good approximation of the OCM for various models and values of a_s
- Good results up to rather high orders also with other expansion functions after softening the first singularities
- The non-power CI expansions give better results than the non-power FO expansions

Optimal choice: non-power, NPPT expansions, which implement the high-order behaviour through the running coupling

x

8.2 α_s from hadronic τ decays

- Input
  - $\delta^{(0)} = 0.2037 \pm 0.0040_{exp} \pm 0.0037_{PC}$ Beneke 2011
  - $c_{1,1}, c_{2,1}, c_{3,1}, c_{4,1} \quad c_{5,1} = 283 \pm 283$ Beneke 2011
  - $\beta_0, \beta_1, \beta_2, \beta_3, \quad \beta_4 = \pm\beta_3^2/\beta_2$ (for error assessment) Pich 2011
- Results:

$$w_{12}: \quad 0.3195 \pm 0.0034_{exp} \pm 0.0031_{PC} \, {}^{+0.0246}_{-0.0137}(c_{5,1}) \, {}^{+0.0018}_{-0.0019}(scale)$$

$$w_{13}: \quad 0.3208 \pm 0.0035_{exp} \pm 0.0032_{PC} \, {}^{+0.0131}_{-0.0093}(c_{5,1}) \, {}^{+0.0024}_{-0.0088}(scale)$$

$$w_{1\infty}: \quad 0.3182 \pm 0.0033_{exp} \pm 0.0031_{PC} \, {}^{+0.0172}_{-0.0111}(c_{5,1}) \, {}^{+0.0025}_{-0.0088}(scale)$$

$$w_{23}: \quad 0.3193 \pm 0.0034_{exp} \pm 0.0031_{PC} \, {}^{+0.0182}_{-0.0115}(c_{5,1}) \, {}^{+0.0023}_{-0.0063}(scale)$$

x

8.3 α_s from hadronic τ decays

- Average of the new determinations:

$$\alpha_s(M_\tau^2) = 0.3195 \pm 0.0034_{exp} \pm 0.0031_{PC}\,{}^{+0.0182}_{-0.0114}(c_{5,1})\,{}^{+0.0018}_{-0.0019}(scale) \pm 0.0005_{\beta_4}$$

- The biggest error is due to the uncertainty of $c_{5,1}$

- Combining errors in quadrature:

$$\boldsymbol{\alpha_s(M_\tau^2) = 0.3195\,{}^{+0.0189}_{-0.0138}}$$

- Comparison: with the same input
- Standard CIPT: $\alpha_s(M_\tau^2) = 0.3419 \pm 0.012$
- Standard FOPT: $\alpha_s(M_\tau^2) = 0.3199^{+0.0118}_{-0.0074}$

- Smaller errors, but the truncation error underestimated

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Outline

x

9.1 CONCLUSIONS

RECAPITULATION: Short history of our method. See also 4.1 - 4.9 and 5.2

- The standard pt expansions in powers of α_s ignore, in every order, the singularities of the expanded function $\widehat{D}(\alpha_s)$. This is paradoxical in view of the fact that the knowledge of the singularities is of importance for the convergence properties of pt expansions and for the determination of $\widehat{D}(\alpha_s)$
- We developed a systematic method of constructing an improved expansion that, besides the four lowest-order expansion coeffs, makes full use of our knowledge of
  - (1)analyticity
  - (2) location and nature of the singularities, and
  - (3) high-order behaviour of the expansion coeffs of the expanded function, which in our case is $B(u)$, the Borel transform of $\widehat{D}(\alpha_s)$.

  See refs [1-4, 6-8] for details.

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9.2 CONCLUSIONS

Every construction calls for specific building constituents which are adjusted to the construction

- Standard Perturbation Theory rests on the following postulates (5.1-5.4):
- 1 Perturbative series is an EXPANSION in POWERS of a parameter
- 2 PERTURBATIVE EXPANSION is CONVERGENT
- 3 The EXPANDED FUNCTION is ANALYTIC in the perturbat.parameter.
- Thanks to Dyson, 3 (analyticity) is lost; singularities exist even at the origin. If one restricts oneself to power series, this implies divergence of the series, ie., 2 (convergence) is lost . As a consequence, Dyson's result implies
- DYSON: Having lost 3 and wishing to retain 1, he LOSES 2
- WE: Having lost 3 and wishing to retain 2, we ABANDON 1
- DYSON: Having lost CONVERGENCE 2, he resorts to asymptotic series. (But asymptotic series lead to fatal AMBIGUITY.)
- WE abandon POWER SERIES 1, extending perturbation theory by using the $\mathcal{W}_n(\alpha_s)$ functions adjusted to the expanded function. (Even within the frame of the different power expansions there exist an infinite amount of perturbative summations.)

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9.3 CONCLUSIONS

Perturbation Theory, RECAPITULATION: STANDARD expansions in the powers α_s^n, u^n

- Expanding the Adler function $\widehat{D}(\alpha_s)$ and its Borel transform $B(u)$:
- 1 IN POWERS of $\alpha_s(s)$ and u, respectively (Standard POWER expansions of $\widehat{D}(\alpha_s)$ and $B(u)$):

$$\clubsuit \widehat{D}(\alpha_s) \sim \sum_{k=0}^{\infty} \alpha_s^{k+1} \Gamma(k+1) \frac{B^{(k)}(0)}{k!} \tag{22}$$

$$\clubsuit B(u) = \sum_0^\infty u^k \frac{B^{(k)}(0)}{k!}$$

- 2 NOW take NON-POWER expansions of $\widehat{D}(\alpha_s)$ (see 2.7, 2.8):

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9.4 CONCLUSIONS. Recall 5.1

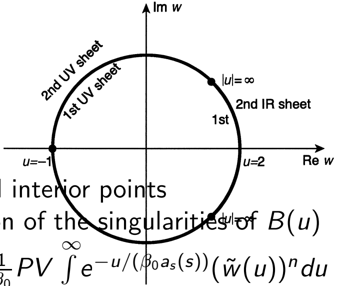
PT, RECAPITULATION: 4 NON-POWER expansions in the singular functions $\mathcal{W}_n^{jk}(s)$, $(\tilde{w}(u))^n$

$$\spadesuit \widehat{D}(s) \sim \sum_n c_n^{(jk)}\, \mathcal{W}_n^{jk}(s)$$

$$\mathcal{W}_n^{jk}(s) = \frac{1}{\beta_0} PV \int_0^\infty \mathrm{e}^{-u/(\beta_0 a_s(s))}\, \frac{(\tilde{w}_{jk}(u))^n}{S_{jk}(u)}\, du$$

- Map the cut u-plane onto the disk $|w| < 1$ in the $w = \tilde{w}(u)$ plane, with $\tilde{w}(0) = 0$
- Non-power expansion of $B(u)$:
  $\spadesuit B(u) = \sum_0^\infty c_n\, w^n, \quad w = \tilde{w}(u)$

$$\boxed{\tilde{w}(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}}$$



- converges in the whole cut u-plane
- best asymptotic convergence rate at all interior points
- full use of our knowledge of the location of the singularities of $B(u)$
- suggests the expansion: $\widehat{D}(s) = \sum_n c_n \frac{1}{\beta_0} PV \int_0^\infty e^{-u/(\beta_0 a_s(s))} (\tilde{w}(u))^n du$

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9.5 CONCLUSION

- Our present work is motivated by the observed discrepancy between the predictions of $\alpha_s(M_\tau^2)$ from the FOPT and CIPT power expansions.
- The results of the new RGSPT expansion are similar to those obtained by the CIPT expansion.
- The dramatic divergence of the perturbative series is tamed by two methods we have applied in the Borel plane: optimal conformal mapping (OCM) and singularity softening (SS).
- It turns out that the renormalon singularities and RG invariance play an important role in perturbation theory.

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Selected list of papers

1. I.C. & J.F., Expansion functions in perturbative QCD and the determination of $\alpha_s(M_\tau^2)$, Phys.Rev.D **84** 054019 (2011), arXiv:1106.5336 [hep-ph]
2. I.C. & J.F., Determination of $\alpha_s(M_\tau^2)$: a conformal mapping approach. Nucl.Phys.Proc.Suppl. **218**:128-133 (2011), arXiv:1011.6480 [hep-ph]
3. I.C, J.F and I. Vrkoč, On the ambiguity of functions represented by divergent power series, Appl. Numer.Math. **60**:1264 (2010), arXiv:1011.6490 [math-ph]
4. I.C. & J.F., α_s from tau decays: Contour-improved versus fixed-order summation in a new QCD perturbation expansion. Eur.Phys.J.C **64**:35-45 (2009),
5. I.C, J.F and I. Vrkoč, On the ambiguity of field correlators represented by asymptotic perturbation expansions, J.Phys.A **42** 395403 (2009)
6. I.C & J.F, Analytic continuation and perturbative expansions in QCD. Eur.Phys.J.C**24**:127-135 (2002)
7. I.C. & J.F., Convergence of the expansion of the Laplace-Borel integral in perturbative QCD improved by conformal mapping. Phys.Rev.D**62**:054007 (2000)
8. I.C. & J.F., Accelerated convergence of perturbative QCD by optimal conformal mapping of the Borel plane, Phys.Rev.D **60**:054014 (1999)

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A P P E N D I C E S

Additional remarks and explanations not presented
in the Oct. 17, 2013 seminar

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2.4 Singular expansion functions

We summarize. Replacing the powers α_s^n by the $\mathcal{W}_n(\alpha_s)$ functions

- 1: Asymptotic series (Dyson '52) imply: 1high ambiguity, 2plenty of information gets lost. The approximants in powers, α_s^n, tell nothing of the singularities of $\widehat{D}(\alpha_s)$. We replace the powers by the $\mathcal{W}_n(\alpha_s)$ functions, whose singularities truly resemble those of $\widehat{D}(\alpha_s)$.
- 2: Using the $\mathcal{W}_n(\alpha_s)$: mild behaviour at $n \to \infty$.
- 3: The τ decay rate into u and d quarks (through a V or an Avector current). The pt QCD contribution is known to $O(\alpha_s^4)$.
- 4: The npt corrections are supposed to be small.
- 5: Main uncertainties: from the h.o.corrections & RG-improvement.
- 6: (3) and (4) are relevant for the extraction of $\alpha_s(M_\tau^2)$. Related to an observable, $\widehat{D}_{pert}(\alpha_s)$ and $\delta^{(0)}$ should be scale independent:

$$\mu^2 \frac{d}{d\mu^2} \widehat{D}_{pert}(\alpha_s) = 0, \qquad \widehat{D}_{pert}(\alpha_s) = \sum_n^{\infty} K_n [a_s(\mu^2)]^n$$

(see 1.2). The OPE contributions are small, pt QCD can be used outside the timelike axis to calculate $\widehat{D}$ along the contour.
- 7: But scale dependence is still present in *TRUNCATED* expansions.

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2.7 Renormalization Group summation

RGSPT

- **RGSPT** (Renormalization Group Summation Perturbation Theory): As suggested by Ahmady & al., the expansion of $\widehat{D}_{FOPT}(a, L)$ is

$$\widehat{D}_{FOPT}(a, L) = \sum_{n=1}^{\infty} a^n D_n(aL), \tag{23}$$

2.6, where the $D_n(v)$, depending only on $v = aL$, are

$$D_n(v) = \sum_{m=0}^{\infty} (m+1) c_{n+m,m+1} v^m \tag{24}$$

- These functions can be obtained in a closed analytical form (Ahmady & al). The Adler function defined by (23) is scale independent:

$$\frac{d}{d\mu^2} \widehat{D}_{FOPT}(aL) = 0 \quad , \qquad \beta(a) \frac{\partial \widehat{D}_{FOPT}}{\partial a} - \frac{\partial \widehat{D}_{FOPT}}{\partial L} = 0. \tag{25}$$

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2.7 Renormalization Group summation

RGSPT

- The first two solutions are

$$D_1(aL) = \frac{c_{1,1}}{y}\ ,\quad D_2(aL) = \frac{c_{2,1}}{y^2} - \frac{\beta_1 c_{1,1} \ln y}{\beta_0 w^2}\quad,\quad y = 1 + \beta_0 aL\ . \qquad (26)$$

- The RGSPT expansion of the Adler function is

$$\widehat{D}_{RGSPT}(aL) = \sum_{n=1}^{N} a^n\ D_n(aL) \qquad (27)$$

defined as infinite series in powers of the variable $v = aL$.

$$\delta^{(0)}_{RGSTP} = \sum_{n=1}^{\infty} a(M_\tau^2)^n\ d_n \qquad (28)$$

$$d_n = \frac{1}{2\pi i} \oint_{|s|=M_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{M_\tau^2}\right)^3 \left(1 + \frac{s}{M_\tau^2}\right) D_n(aL). \qquad (29)$$

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4.2 Renormalization-group summation: FOPT, CIPT

The standard pt expansion of $\widehat{D}$ in a definite Rscheme, denoted FOPT [BJ], is

$$\widehat{D}_{FOPT}(\alpha_s) = \sum_{n\geq 1}(a_s(\mu^2))^n \ \ [c_{n,1} + \sum_{k=2}^{n} k\ c_{n,k}\ (\ln\frac{-s}{\mu^2})^{k-1}] \qquad (30)$$

see 1.2. The Rscale μ^2 is chosen close to s_0, the $c_{n,1}$ are calculated from Feynman diagrams. The $c_{n,k}$ for $2 \leq k \leq n$ depend on the $c_{n,1}$ and on the β_k of the RG β-function, presently known to 4 loops.
In the $\bar{MS}$ scheme for $n_f = 3$, the coeffs $c_{n,1}$, $n = 1, 2, 3, 4$ calculated up to now are 1, 1.64, 6.371, and 49.079 resp. By setting $\mu^2 = -s$ in (30), one obtains the RG improved or contour improved perturbation theory (CIPT):

$$\widehat{D}_{CIPT}(\alpha_s) = \sum_{n\geq 1} c_{n,1}(a_s(-s))^n \qquad (31)$$

where the running coupling $a_s(-s)$ is determined by solving the RG equation

$$s\ \mathrm{d}a_s(-s)/\mathrm{d}s = \beta(a_s). \qquad (32)$$

.
This equation is solved along the inrtegation contour $|s| = s_0$, from the input $a_s(-s)$ at $s = -M_Z^2$.

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4.3 Renormalization-group summation: RGSPT

This equation is solved numerically, iteratively along the integration contour $|s| = s_0$, starting from the input $a_s(-s)$ at $s = -M_\tau^2$. The above expansions (convergence, behaviour in the s-plane) have been critically examined Davier08,BJ08,Pich11,BJ/ed.Bethke11,BeBoiJa13. New prescription Ahmady02,03. Generalization of leading logs to nonleading ones, by summing all terms obtained from RG invariance. AACF13. (30) can be written as

$$\widehat{D}_{RGSPT}(\alpha_s) = \sum_{n\geq 1} (\tilde{a}_s(-s))^n \;\; [c_{n,1} + \sum_{j=1}^{n-1} c_{j,1}\; d_{n,j}(y)], \qquad (33)$$

where

$$\ln \tilde{a}_s(-s) = \tilde{a}_s(\mu^2)/[1 + \beta_0 a_s(\mu^2) \ln(-s/\mu^2)] \qquad (34)$$

is the solution of the RGE 32 to one loop.

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4.4 Other high-order effects

- Abbas,Ananthanarayan,Caprini: In [AbAnCa], the perturbative expansion is improved by using the leading logarithms obtained from

  RENORMALIZATION GROUP INVARIANCE (RGS),
  see [MMA,Ahmady].

  remarkably small.

- Confrontation of the CIPT summation with the FOPT one improved by RGS (see 2.8). The difference between the CIPT and the RGS- improved FOPT sums is REMARKABLY SMALL. Thus, the CIPT summation in the NPPT expansion has, with great accuracy, a very similar effect as the FOPT improved by RGS. This signals the need of extra information inputs, additional to the values of the lowest-order four expansion coeffs. The RG effects of the CIPT and RGS approaches overlap here, they percolate.

AACF : Combining RGS with NPPT (ie., large order behavior by means of OCM and SS). There are two sources of ambiguity, RG and NPPT expansion.

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Outline

4.5 A NEW EXPANSION of $\widehat{D}(\alpha_s)$ is needed

- **The singularities of $\widehat{D}(\alpha_s)$, if encoded in renormalons at FINITE distance from $u = 0$, are located AT $\alpha_s = 0$.**
- **A serious defect of Standard PT:** The singular $\widehat{D}(\alpha_s)$ is expanded in the α_s^n, which, analytic in α_s, *DO NOT SEE* the physics hidden in the singularities of $\widehat{D}$. So, some ESSENTIAL PHYSICS may be INVISIBLE to standard PT. (L'essentiel est invisible pour la theorie des perturbations.) Each approximant is analytic, in contrast to $\widehat{D}(\alpha_s)$.
- THE FORM of our expansion functions depends on which property of $\widehat{D}$, a physical quantity, is considered important: symmetry, analyticity, singularities, etc. In keeping with it, the task is to construct expansion functions having the same property. Thus,

  ♣ **Every construction calls for specific building constituents**,
  ♣ **Every function calls for specific expansion functions**
  See also 9.1 - 9.5.

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4.6 A NEW EXPANSION of $\widehat{D}(\alpha_s)$ is needed

- Apply this to the *SINGULAR* $\widehat{D}(\alpha_s)$. Expand $\widehat{D}(\alpha_s)$ in a set of functions whose singularities resemble those of $\widehat{D}(\alpha_s)$ as much as possible:
- **Expansion of a SINGULAR function** $\widehat{D}(\alpha_s)$ calls for **SINGULAR expansion functions**, $\mathcal{W}_n(\alpha_s)$ (*).
- NOTE: The Standard PT does NOT observe the demand (*).
  Indeed, $\widehat{D}(\alpha_s)$ is **SINGULAR**,
  while the expansion functions are α_s^n, i.e., **ANALYTIC** .
- This is why the powers α_s^n are not suited for expanding the singular $\widehat{D}(\alpha_s)$. We build up
  Non-Power Perturbation Theory (NPPT),
  in which the rule (*) is observed (see 8.1, 10.4).

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4.7 APPLY (*) to $B(u)$, rather than to $\widehat{D}(\alpha_s)$

- INSTEAD of expanding ♠ *ADLER FUNCTION* $\widehat{D}(\alpha_s)$ in α_s^n,
  we can expand **its ♠ BOREL TRANSFORM** $B(u)$ **in** u^n
- *WHY* do we do so? *WHAT IS BETTER*, and why? The expansion of $\widehat{D}$ and B behaves like $n!\alpha_s^{n+1}$ and u^n respectively. In more detail:
- *REMIND* Watson's lemma. You have (page 3.2) two series:

$$B(u) = \textstyle\sum_0^\infty u^k B^{(k)}(0)/k!$$

$$\widehat{D}(\alpha_s) \sim \sum_{k=0}^{\infty} \alpha_s^{k+1} \Gamma(k+1) \frac{B^{(k)}(0)}{k!}.$$

Note the factor $\Gamma(k+1)$. We use the formula

$$\mathbf{1}/\rho_\mathbf{B} = \limsup_{\mathbf{n}\to\infty} \sqrt[n]{|a_n|},$$

for $1/\rho_B$, and $1/\rho_D = \limsup_{n\to\infty} \sqrt[n]{n!|a_n|}$ for $1/\rho_D$, where ρ_B and ρ_D is the convergence radius of the two series respectively. If ρ_B is finite nonvanishing ($=1$ for the UV renormalon), ρ_D is **zero, and the expansion of $\widehat{D}(\alpha_s)$ diverges for every $\alpha_s \neq 0$**.

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4.8 APPLY (*) to $B(u)$, rather than to $\widehat{D}(\alpha_s)$

4 Expansion of a *SINGULAR* function $B(u)$ calls for *SINGULAR* expansion functions, $(\tilde{w}(u))^n$

- Unlike in the α_s-plane, the boundary of $\mathcal{H}$ in the u-plane NEVER TOUCHES the orign. The expansion of $B(u)$ in u^n has much a milder h.o. behaviour than that of $\widehat{D}(\alpha_s)$ in $\alpha_s s^n$ (factor $n!$). There are two steps of improvement:
- STEP 1: Do NOT EXPAND $\widehat{D}(\alpha_s)$ in α_s^n,
but rather **EXPAND** $B(u)$ in u^n. *HOWEVER*,
- STEP 2 is an improvement of Step 1: EXPAND $B(u)$ **NOT in** u^n**,** *but in the* $(\tilde{w}(u))^n$ **(OCM, optimal conformal mapping),** $\tilde{w}(u)$ **having the same location of singularities as** $B(u)$.
- *CONCLUSION*: Rather than in u^n, expand $B(u)$ in the $(\tilde{w}(u))^n$, since $\tilde{w}(u)$ maps $\mathcal{H}$ onto the DISK $|\tilde{w}(u)| \leq 1$. *INTUITIVELY*: As $\tilde{w}(u)$ and $B(u)$ have identical location of singularities, the expansion of B in the $\tilde{w}^n$ is simpler and easier to handle than that in the u^n. EXACTLY sec. 6. To make full use of analyticity,

USE OCM in the Borel plane.

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4.9 OCM makes $\mathcal{H}$ COINCIDE with the convergence disk

see 4.11, 6.5 and the sections 5 and 6

- ♣ Let $\mathcal{H}$ be the analyticity domain of $B(u)$ in the u-plane,
  ♣ Let $\mathcal{G}_w$ be a subset (region) of $\mathcal{H}$: $\mathcal{G}_w \subset \mathcal{H}$,
  ♣ Let $w(u)$ map $\mathcal{G}_w$ conformally onto $|w(u)| < 1$
  ♣ Let $\tilde{w}(u)$ map $\mathcal{G}_{\tilde{w}} = \mathcal{H}$ conformally onto $|\tilde{w}(u)| < 1$
- In QCD, the singularities of $B(u)$ are located along the rays $u > 2$ and $u < -1$ (IR and UV renormalons). $\mathcal{H}$ is the u-plane cut along the rays; see 4.10 and 6.5 for details
- IF the power series of a function analytic in u is DIVERGENT at some $u \in \mathcal{H}$, convergence is **restored by OCM at that point**.
- IF the power series is CONVERGENT at some $u \in \mathcal{H}$, the convergence rate is **enhanced by OCM at that point**.
- This is impossible in the α_s-plane, where the point $\alpha_s = 0$ *MAY APPEAR ON THE BOUNDARY* of the analyticity domain. See 4.8. This is why **$B(u)$ should be preferred to $\widehat{D}(\alpha_s)$, and Expansions of $B(u)$ in $(\tilde{w}(u))^n$ should be preferred to those in u^n**.

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4.10 Discussion

See also 2.10, 4.5, 4.6, 6.5 and sec. 5 for details

- Take $\mathcal{G}_w \subset \mathcal{H}$. If $\mathcal{G}_w = \mathcal{H}$, we write $\tilde{w}(u)$, $\mathcal{G}_{\tilde{w}}$. The equality $\mathcal{G}_{\tilde{w}} = \mathcal{H}$ implies that the whole doubly-cut u-plane (which is simply-connected) is mapped onto the $|\tilde{w}(u)| < 1$ disk, $\tilde{w}(u)$ having the same *LOCATION* of singularities as $B(u)$. The disk $|\tilde{w}(u)| < 1$ is the

  DISK OF CONVERGENCE of the expansion of $B(u)$ in the $(\tilde{w}(u))^n$,
  and IS IDENTICAL
  with $\mathcal{H}$, the REGION OF ANALYTICITY of $B(u) = B(u(\tilde{w}))$.

- See **Lemma 1** and **Lemma 2**, sec. 6

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5.2 More about conformal mapping method

See also 11.7

- Soon after Ciulli,JF,Nucl.Phys.1961, similar methods of improving conver-gence properties of infinite series appeared . However, only our paper contains proofs of the properties of OCM. Our result remained unnoticed for decades; we therefore devote a discussion of it in ref. [1] and here in sec. 6.
- The function $w(u)$ conformally maps the region $\mathcal{G}_w \subset \mathcal{G}_{\tilde{w}} = \mathcal{H}$ in the u-plane onto the disk $|w(u)| < 1$. Using $w(u)$, we obtain a series with better convergence properties than the expansion in u^n.
- What are the merits and virtues of OCM ($w = \tilde{w}$) in applications?

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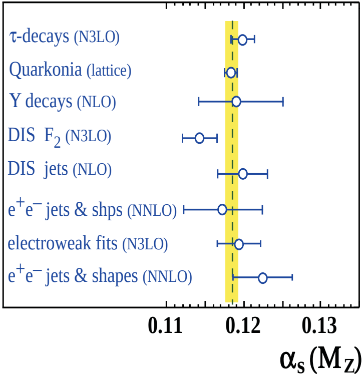
9.2 Status of the α_s determination

From all processes that involve gluons



S. Bethke et al, Summary of α_s measurements, arXiv:1110.0016 [hep-ph]:

World average 2011: $\alpha_s(M_Z^2) = \mathbf{0.1183 \pm 0.0010}$

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10.2 CONCLUSIONS

An example: 2 Resemblance in symmetry

- EXAMPLE: Partial wave expansion of $A(s,t)$, the elastic scattering amplitude of two spinless particles:

$$A(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta). \tag{35}$$

Here, s and t is the c.-m. energy and the momentum transfer squared respectively, while $a_l(s)$, θ, and P_l is the partial wave amplitude, the scattering angle and the Legendre polynomial of l-th degree, respectively. NOTE: one could use many other function sets to expand the angular dependence of $A(s,t)$, but (35) is preferred, exhibiting rotational symmetry, (SO(3)).

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9.6 CONCLUSIONS; see also 5.1 - 5.7

Non-Power PT (NPPT), RECAPITULATION: 8 Replacing the powers α_s^n by the $\mathcal{W}_n(\alpha_s)$

- The standard pt expansion functions (powers of α_s) are replaced by the $\mathcal{W}_n(\alpha_s)$, which are singular at $\alpha_s = 0$ and have divergent expansions in the α_s^n, thereby resembling, as far as our knowledge allows, the expanded Adler function $\widehat{D}(\alpha_s)$ itself.
- If $\alpha_s \to 0_+$, the functions $\mathcal{W}_n(\alpha_s)$ tend to zero with the same rate: $\mathcal{W}_n(\alpha_s) \sim (\alpha_s)^n$.
- In the CI version, the new expansion provides a solid theoretical frame for a precise determination of $\alpha_s(M_\tau^2)$ from τ decays.
- The OCM $\tilde{w}(u)$ allows one to expand $B(u)$ in powers of $\tilde{w}(u)$, a function that has the same location of singularities as $B(u)$. This expansion has the fastest convergence rate.
- Several choices of "singularity softening" and conformal mappings give consistent results up to high orders.

9.7 CONCLUSIONS; see also 4.1 - 4.7

- At first sight, the difference between the standard and the non-power perturbation theories consists in the difference between the two methods of approximating the solution of the equation of motion of the interaction in question. Note, however, that nothing is known about the relation between the exact solution of the equations of motion and its approximation by means of the standard perturbation power expansion on one side and the non-power perturbation expansion on the other side. While the former approximation is based on the well-known idea that the interaction is mediated by the exchange of one, two, three, four, etc gauge bosons of the relevant interaction (thereby being based on a mechanical model of the particle exchange between the interacting objects according to the corresponding of the interaction order), the idea of the latter approximation is based on the universal assumption that the fundamental physical properties are encoded in the singularities of the functions that describe the interaction.

9.8 CONCLUSIONS; see also 4.1 - 4.7

- Our results published during the past 10-15 years on this subject indicate that the description based on the singularities is more appropriate for the description of the interaction processes and comes nearer the truth and reality of Nature. // gives a truer picture of the reality of Nature// occurring in Nature. It is not an empty matter-of-course that, for small enough values of the coupling parameter, the two different mathematical approaches which are based on two markedly different philosophies coincide both numerically and philosophically.

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9.9 CONCLUSIONS

Perturbation Theory: 9 Merits and virtues of our approach

- The graphs on the pages 9.5-9.7 show how the standard expansion (left) and the non-power one (right) describe the model of Beneke and Jamin at high orders. The difference is apparent above $N = 10$, i.e., out of reach of our present knowlege of the coeffs. But even in this situation, one can estimate what is preferred when the standard expansion and when the non-power one is used.
- The results show how useful it is to account for singularities of $\widehat{D}(\alpha_s)$, in spite of our scarce knowledge of them: both the CI and the FO expansions tend considerably faster to the exact value in our non-power expansions, in which the singularities of $\widehat{D}$ are accounted for. Violent oscillations typical for the *standard* expansions do not appear in the non-power ones, **the singularities being deposited in the $\mathcal{W}_n(\alpha_s)$ functions,** while the powers α_s^n of the standard expansion fully ignore them. The non-power expansion prefers CIPT to FOPT, which can be understood in view of the fact that RG invariance is taken into account in CIPT; see [1,2,4].

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10.5 CONCLUSIONS

Perturbation Theory, RECAPITULATION: 5 Our approach: Resemblance in singularities

- **NOVEL APPROACH to perturbation theory (PT)**
- **Standard PT:** Adler function has two parts, one describing the free system and the other one describing interaction. The latter part is an infinite series of powers of the coupling. Till 1952, the convergence of pt series had been universally adopted. In 1952 F. Dyson published a proof that the pt series in QED are divergent. This raised the problem of expansions having terms dramatically growing with growing order. This reveals strong singularities in the coupling complex plane, and calls for NON-POWER Perturbation Theory:
- **Replacing the standard powers by special functions** that have the same (or similar to, according to our knowledge) singularities (location, nature) as the expanded function.

10.6 CONCLUSIONS

Perturbation Theory, RECAPITULATION: 6 Our approach

- Our expanding functions (denoted generically $\mathcal{W}_n(\alpha_s)$) should implement all information about the singularities of the expanded function. Then, the terms are not powers of the coupling, the corections being big at low orders, decreasing with growing order.
- The methods applied here to pt theory were developed in 1960 as a part of the program of the analytic theory of scattering amplitudes, where problems of the determination of the S matrix were examined. S.Ciulli, J.Fischer, Nucl.Phys.**24**,465(1961)

9.1 CONCLUSIONS

Perturbation Theory: 1 Numerical approximation

- Numerical agreement is not the only aspect of approximation. There are other properties (symmetries, singularities) which, if contained by the expanded function, are expected to be present in each term as well. Each approximant is supposed to have analogous properties as the expanded function; see 4.2-4.5, 5.3

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9.5 CONCLUSIONS

RECAPITULATION: Short history of our method. See also 4.1 - 4.7 and 5.2

- The standard pt expansions in powers of α_s ignore, in every order, the singularities of the expanded function $\widehat{D}(\alpha_s)$. This is paradoxical in view of the fact that the knowledge of the singularities is of importance for the convergence properties of pt expansions and for the determination of $\widehat{D}(\alpha_s)$
- We developed a systematic method of constructing an improved expansion that, besides the four lowest-order expansion coeffs, makes full use of our knowledge of
  - (1)analyticity
  - (2) location and nature of the singularities, and
  - (3) high-order behaviour of the expansion coeffs of the expanded function, which in our case is $B(u)$, the Borel transform of $\widehat{D}(\alpha_s)$. See refs [1-4, 6-8] for details.

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9.10 CONCLUSIONS; see also 4.1 - 4.7

Every construction calls for specific building constituents which are adjusted to the construction

- Standard Perturbation Theory rests on the following postulates (5.1-5.7):
- 1 Perturbative series is an EXPANSION in POWERS of a parameter
- 2 PERTURBATIVE EXPANSION is CONVERGENT
- 3 The EXPANDED FUNCTION is ANALYTIC in the perturbat.parameter.
- Thanks to Dyson, 3 (analyticity) is lost; singularities exist even at the origin. If one restricts oneself to power series, this implies divergence of the series, ie., 2 (convergence) is lost . As a consequence, Dyson's result implies
- DYSON: Having lost 3 and wishing to retain 1, he LOSES 2
- WE: Having lost 3 and wishing to retain 2, we ABANDON 1
- DYSON: Having lost CONVERGENCE 2, he resorts to asymptotic series. (But asymptotic series lead to fatal AMBIGUITY.)
- WE abandon POWER SERIES 1, extending perturbation theory by using the $\mathcal{W}_n(\alpha_s)$ functions adjusted to the expanded function. (Even within the frame of the different power expansions there exist an infinite amount of perturbative summations.)