RELATIVISTIC METHODS FOR SATELLITE NAVIGATION

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Phys. Rev. D 106, 044034 (2022), github.com/justincfeng/squirrel.jl, github.com/justincfeng/cereal.jl

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MINISTRY OF EDUCATION, YOUTH AND SPORTS

Motivation

- Practical application for general relativity: satellite navigation (GNSS[1])
  - Time dilation from GR & SR can lead to accumulated errors[2] $\sim 10\ \mathrm{km/day}$
- GNSS accounts for relativity through timing corrections; framework is "Newtonian"
- Why consider a fully relativistic approach to GNSS?
  - Theoretically appealing
    - Of fundamental interest too---concerns a class of coords. for spacetime
  - "Newtonian" approach is really SR in disguise[3]
  - Can reduce required ground infrastructure and improve performance[4]

1. Global Navigation Satellite Systems
2. N Ashby, LRR 6 43 2003
3. Ramón Serrano Montesinos, Juan Antonio Morales-Lladosa, Universe 2024, 10(4), 179
4. Carloni et al GRG, Vol. 52, No. 2, (2020)

RPS: Basic idea

- Consider a $1+1$ spacetime, with 2 satellites A and B, broadcasting timestamps with proper times τ_{A} and τ_{B}
- Each value of τ_{A} and τ_{B} defines light cones
- Surfaces can be used to coordinatize a region of spacetime; intersection of light rays identify a point X_c^μ
- Can generalize to $3+1$ dims with 4 satellites:
    - Use proper times $(\tau_1, \tau_2, \tau_3, \tau_4)$ as a local coordinate system (emission coords. [1])
- Fundamental limits on proper time precision place limits on distinghuishability of spacetime points

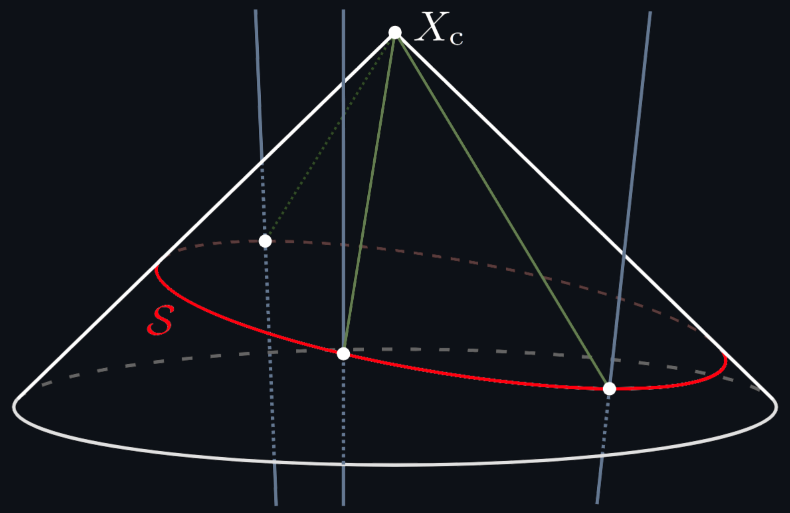
RPS in $1+1$ dims. Time direction is upwards, $c=1$ (light travels on 45° lines)

1. Coll, Pozo, CQG 23:7395 2006; Tarantola et al (2009) (arxiv:0905.3798)

Relativistic location problem

- Emission coords. give instant position, but...
- Have to translate emission coords. to standard coords. for X_c^μ
    - Given satellite trajectories $X_I^\mu(\tau_I)$, find intersection of null geodesics
- A bit complicated in flat spacetime, need to solve a quadratic:

$$\eta_{\mu\nu}(X_c^\mu - X_I^\mu)(X_c^\nu - X_I^\nu) = 0$$



Past light cone of user reception event. For flat $d+1$ spacetime, d emission points define a conic section

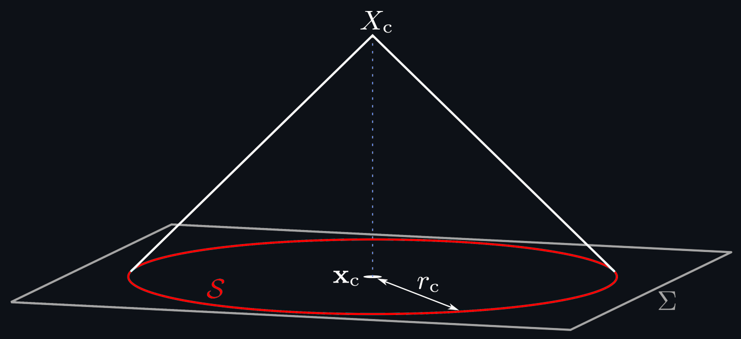
Relativistic location in flat spacetime

- A full solution for the following in [1]:

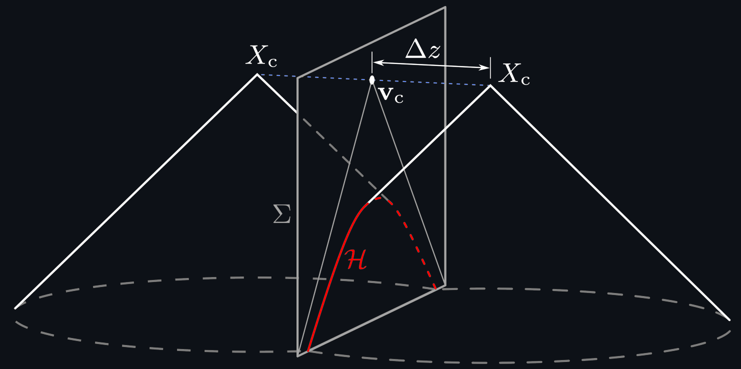
$$\eta_{\mu\nu}(X_c^\mu - X_I^\mu)(X_c^\nu - X_I^\nu) = 0$$

- An intuitive approach outlined in [2]:
  - Lorentz transform so emission points lie on plane
  - Timelike case: circumscribe 2-sphere
  - Spacelike case, circumscribe 2-hyperboloid
- For 4 emission points X_I, have two solns.
  - Can avoid with 5 emission points [3]:

$$\left[2X_c^\mu\left(X_J^\nu - X_I^\nu\right) + X_I^\mu X_I^\nu - X_J^\mu X_J^\nu\right]\eta_{\mu\nu} = 0$$



If emission events are on a spacelike surface



If emission events are on a timelike surface

1. Coll, Ferrando and Morales-Lladosa, CQG 27 065013 (2010)

2. JCF, F. Hejda, S. Carloni PRD 106, 044034 (2022) https://github.com/justincfeng/cereal.jl

3. M. L. Ruggiero, A. Tartaglia, and L. Casalino, Adv. Space Res. 69 4221 (2022)

Do we need curved spacetime?

- Gravity needed for time dilation, but for relativistic location, curved spacetime effects are small $\sim 1\ \mathrm{cm}$
- With flat spacetime formulas, can have large errors due to atmosphere and ionosphere
  - Can model with Gordon metric [1]:

    $$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \left(1 - n^{-2}\right) u_\mu u_\nu$$

    where n is index of refraction, u^μ is four-velocity.
- Have numerical methods for relativistic location problem in curved spacetime [2,3]

1. W. Gordon, Annalen der Physik 377, 421–456 (1923); C. Barcelo, S. Liberati, and M. Visser, Living Rev. Rel.8, 12 (2005), arXiv:gr-qc/0505065
2. D. Bunandar, S. Caveny, R. A. Matzner, PRD 84, 104005 (2011)
3. JCF, F. Hejda, S. Carloni Phys. Rev. D 106, 044034 (2022) arXiv:2201.01774

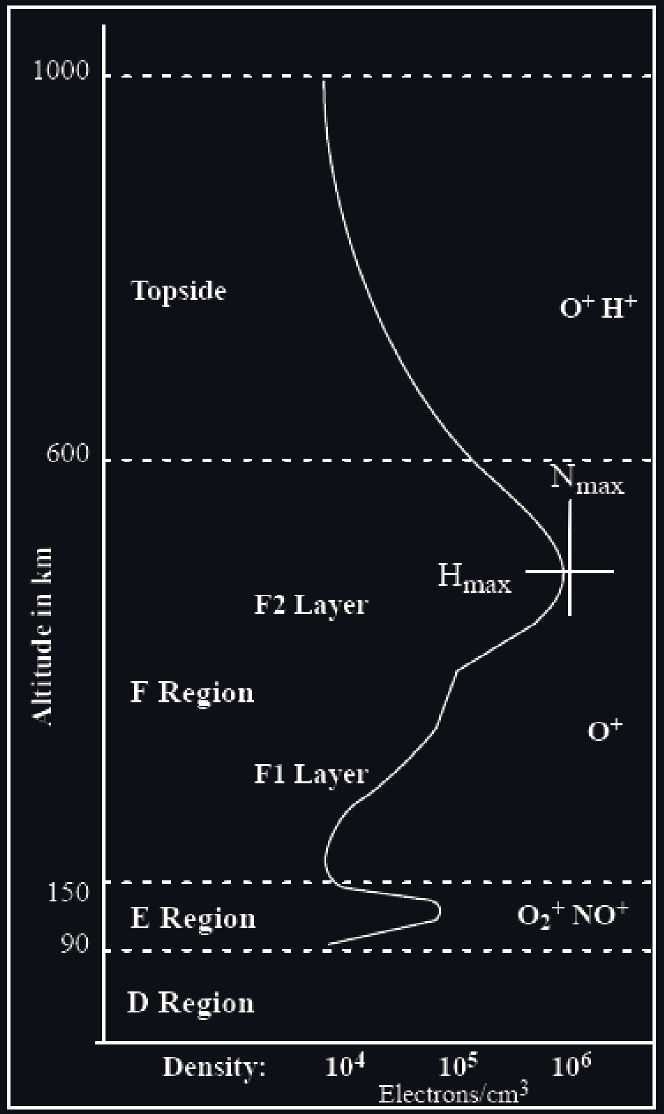
Ionospheric index of refraction

- For GPS signals, ionosphere and troposphere are dominant contributions to delay
- Ionosph. group index of refraction ($f_{\mathrm{GNSS}} \sim 1$ GHz):

$$n_{\mathrm{ion}} \approx 1 + \frac{\omega_{\mathrm{p}}^2}{f^2} = 1 + \left(4.024 \times 10^{-11}\right)[N_{\mathrm{e}}/\mathrm{cm}^{-3}]$$

  $\omega_{\mathrm{p}} \sim 10$ MHz is electron plasma freq.

- Index of refraction roughly isotropic if electron gyrofrequency ω_{g} is small
  - $\omega_{\mathrm{g}}/(2\pi f) \sim 3 \times 10^{-3}$ for Earth's $\vec{B}$ field
- Peak value $\Delta n_{\mathrm{ion}} := n_{\mathrm{ion}} - 1 \sim 4 \times 10^{-5}$

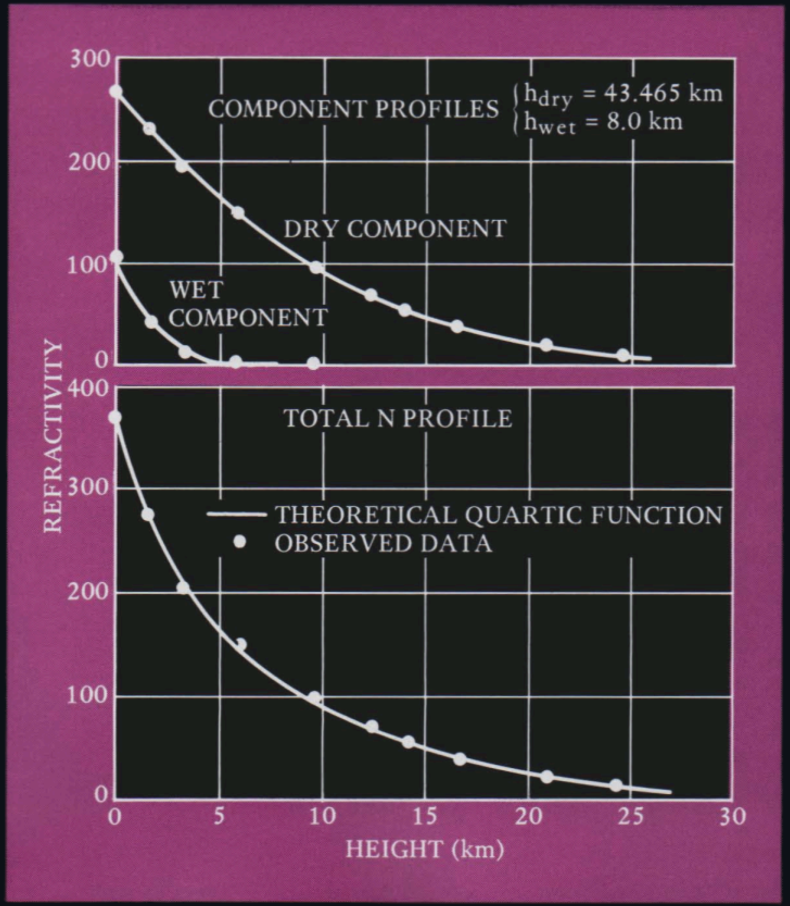


NOAA: http://www.sel.noaa.gov/info/Iono.pdf

Can obtain Eqns from approximation to Appleton-Hartree equation.

Tropospheric index of refraction

- For the troposphere, index of refraction depends on temperature, pressure, and humidity
  - Modeled using the Edlén Equation [1,2]
- Monotonically decreases with height
- Peak value: $\Delta n_{\mathrm{atm}} \sim 4 \times 10^{-4}$
- For our tests, we assume a simple profile
  - Realistic profiles may require inference
  - Adaptive optics / wavefront sensors?



Refractivity $10^6 \Delta n_{\mathrm{atm}}$ of troposphere vs. height [Fig 6, Hopfield, APL Technical Digest 11 (1972).]

1. K. P. Birch and M. J. Downs, Metrologia 30, 155 (1993)
2. B. Edlén, “The Refractive Index of Air,” Metrologia 2, 71–80 (1966).

Relativistic location in curved spacetime

- We write function $\mathrm{x}^\mu(\underline{X}_e, v)$ to find geodesic endpoints numerically from:

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$

- For four emission points, constraint is $F(\underline{X}_e, v) = 0$, with

$$F(\underline{X}_e, v) := (\mathrm{x}_1 - \mathrm{x}_2, \mathrm{x}_1 - \mathrm{x}_3, \mathrm{x}_1 - \mathrm{x}_4)$$

  Can compute Jacobian of $f(v) = F(\underline{X}_e, v)$ using *automatic differentiation*

- `squirrel.jl` algorithm [1]:

  i. Use flat spacetime algorithm to make guess for initial 3-velocity v

  ii. Find roots of $f(v) = F(\underline{X}_e, v)$ to obtain corrected initial velocity v

  iii. Integrate geodesics with corrected v to find intersection point P

1. https://github.com/justincfeng/squirrel.jl

Automatic differentiation: *"Stop approximating derivatives!"*

- Refers to efficient methods for machine-precision derivatives.
- Julia language has good libraries for AD [1]
  - **Can use to differentiate numerical ODE solutions!**
- Forward mode AD [2]
  - Dual numbers: "like complex numbers, but with a 1D Grassman number $\{\varepsilon : \varepsilon^2 = 0\}$ in place of i"
  - For analytic $f(x)$: $f(x + \varepsilon y) = f(x) + \varepsilon y f'(x)$
  - Forward AD easy to implement (in single tweet!)
    ∃ Libraries in Julia, Fortran, C

Mark Saroufim
@marksaroufim

AD in #JuliaLang by @DavidPSanders

```
struct D <: Number
    p
    d
end
import Base: +, *, ^
+(a::D, b::D) = D(a.p + b.p, a.d + b.d)
*(a::D, b::D) = D(a.p * b.p, a.p * b.d + a.d * b.p)
*(b::Real, a::D) = D(b * a.p, b * a.d)
∂(f, x) = f(D(x, 1)).d

julia> ∂(x -> x^2 + 2x, 3)
8
```

6:44 PM · Sep 5, 2020 · Twitter Web App

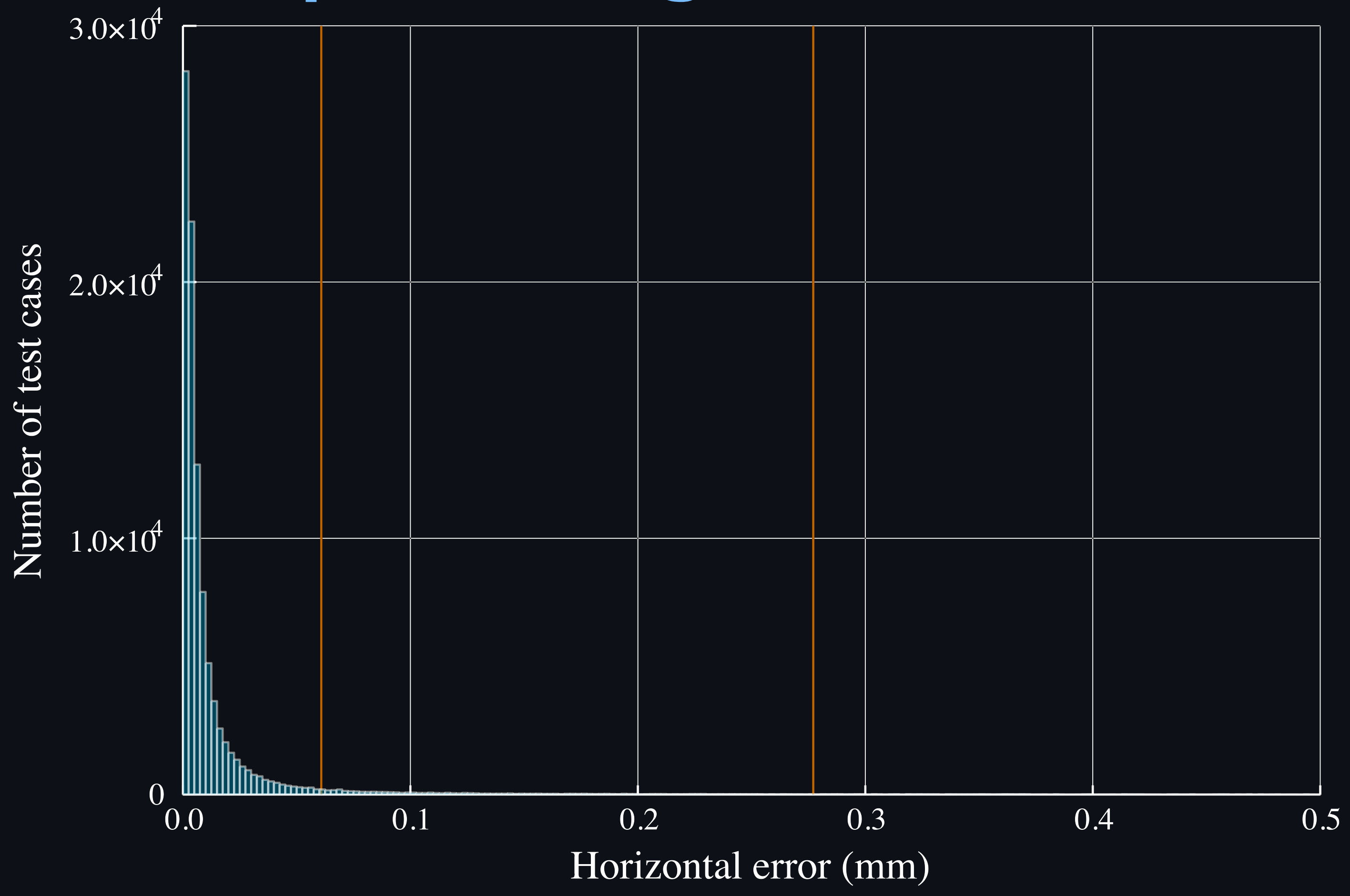
https://twitter.com/marksaroufim/status/1302301588925472768

[1] https://juliadiff.org/
[2] D Austin, *How to Differentiate with a Computer*, AMS column: http://www.ams.org/publicoutreach/feature-column/fc-2017-12

What is AD good for?

- AD can differentiate complicated routines---even numerical ODE solvers!
  - Used to compute Jacobians for geodesics in `squirrel.jl`.
- Can open up new approaches to constructing and solving differential equations.
  - Can provide more efficient alternative to shooting methods
  - Can be used in place of finite difference templates in ODEs/PDEs
  - Can construct EoMs directly from Lagrangian!

Results: Horizontal positioning, vacuum (Kerr-Schild)

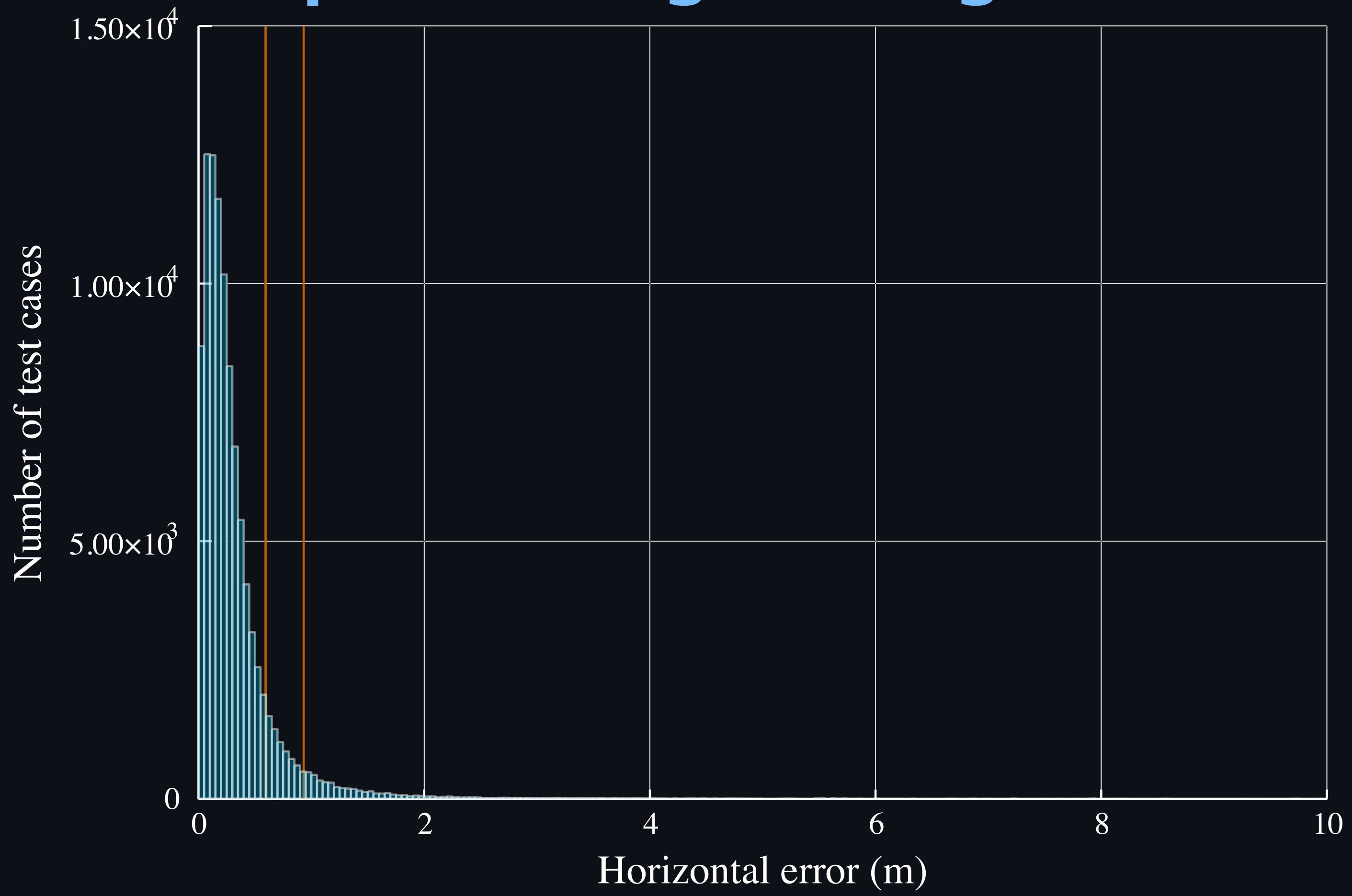


$$\text{rms} = 0.28\text{mm}$$

$$95\%\text{cl} = 0.061\text{mm}$$

JCF, F. Hejda, S. Carloni Phys. Rev. D 106, 044034 (2022) arXiv:2201.01774

Results: Horizontal positioning, Analogue model



$\mathrm{rms} = 0.59\mathrm{m}$

$95\%\mathrm{cl} = 0.93\mathrm{m}$

With 10% uncert. in ionospheric profile

Results: Horizontal positioning errors for Galileo GNSS

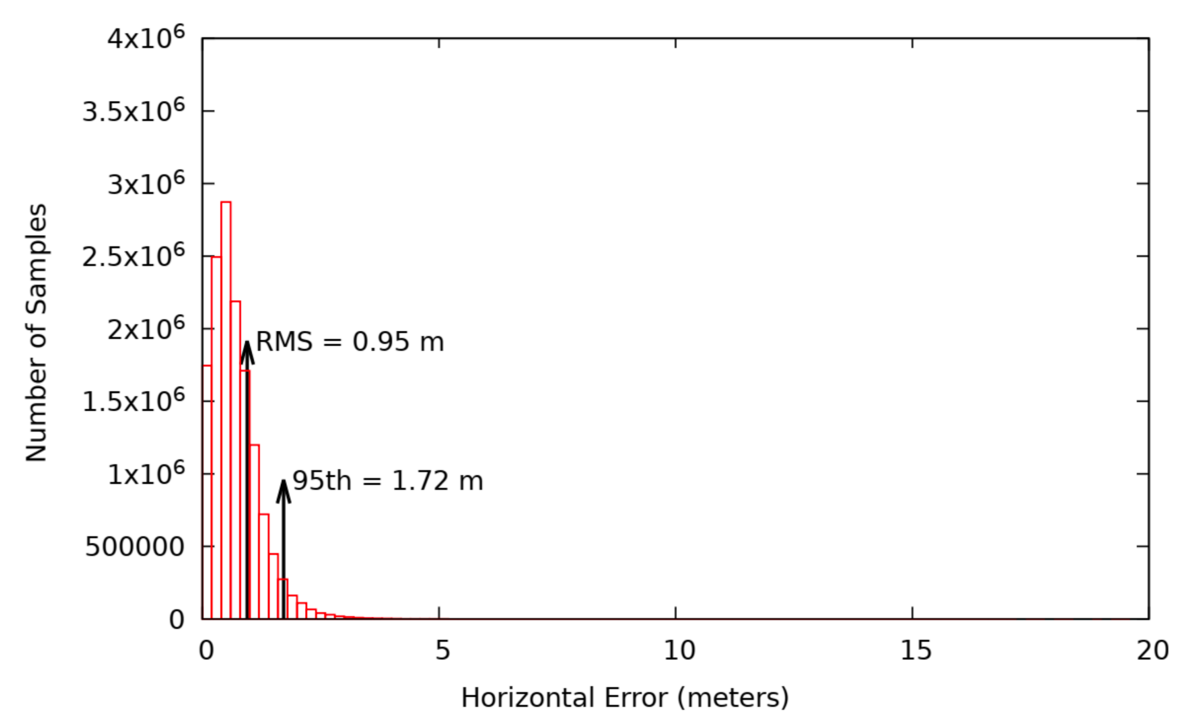
March 2021

HPE

3D Positioning

E1-E5a (DF)

F/NAV



Note: 155 samples > 20 [m], less than 0.01 %

Fig. 21, European Union, "European GNSS (Galileo) Services Open Service Quarterly Performance Report Jan–Mar 2021." (2021)

What next?

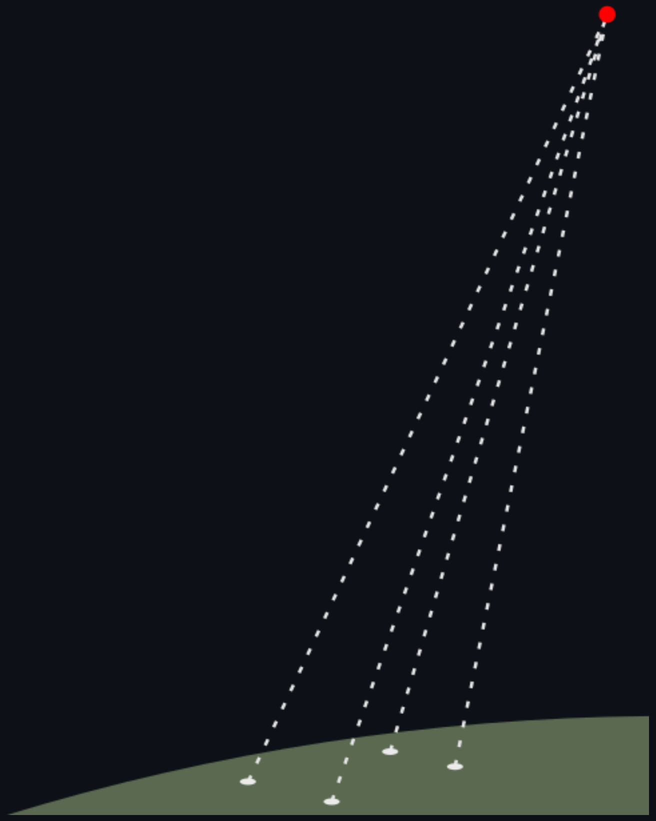
- Problem: Need to know satellite trajectories & ionospheric/atmospheric profiles
    - Need extensive ground tracking infrastructure
    - Need to model ionosphere & atmosphere; space weather events disruptive
- Given a satellite constellation and many users, can one "bootstrap" the problem?
    - Simple function counting arguments suggest that there are enough constraints to bootstrap positions and constrain metric parameters
- Grand vision:
    - GNSS that can simultaneously track satellites, infer ionospheric/atmospheric profiles, and locate users in real time with a few reference stations

What I'm working on now: The inverse problem

- Inverse problem
  - Given a set of users, can you infer the location of the emitter?
  - Idea: run relativistic location in reverse
  - Possible to do even better---can infer index of refraction with large number of users
  - Preliminary tests indicate this is possible
- Current work: proof of concept
- Flat-space Gordon metric:

$$ds^2 = -\tilde{N}^2 dt^2 + dx^2 + dy^2 + dz^2$$

Symmetric $\tilde{N}$ can yield analytic solutions for geodesics



Future directions & vision

- Long-term vision for GNSS design
  - Constellation of simple satellites broadcasting proper times
  - Large user base, track satellites & infer model params / n in real time
  - Goal: GNSS robust to disruption by terrestrial and solar weather events
- Space navigation
  - Navigation near the moon w/ GNSS satellites
  - Solar wind can affect signal propagation on AU scales
  - Analog gravity models may be necessary for interplanetary navigation

Flat Earth satellite tracking

- Flat-Earth symmetry:

$$ds^2 = -dt^2/N(z)^2 + dx^2 + dy^2 + dz^2$$

- Null geodesic equation:

$$-N(z)^2 e^2 + v_{\mathrm{h}}^2 + v_{\mathrm{z}}^2 = 0.$$

- Symmetries yield consts. of motion:

$$e = N(z)^{-2}\frac{dt}{d\lambda}, \qquad v_{\mathrm{h}} := \sqrt{v_{\mathrm{x}}^2 + v_{\mathrm{y}}^2}$$

- Can obtain equation for z:

$$v_{\mathrm{z}} := \frac{dz}{d\lambda} = \sqrt{N(z)^2 e_{\mathrm{t}}^2 - v_{\mathrm{h}}^2}$$

- Expand $N(z) = 1 + \varepsilon\delta N(z)$:

$$\kappa\Delta\lambda \approx \int_{z_0}^{z} dz' \left[1 - \mathcal{E}^2 \varepsilon\delta N(z')\right].$$

where: $\kappa := \sqrt{e_{\mathrm{t}}^2 - v_{\mathrm{h}}^2}, \qquad \mathcal{E} := e_t/\kappa.$

- To $\mathcal{O}(\varepsilon)$, can solve for general $\delta N(z)$
- Higher order needed for ionosphere?
  - For Gaussian $\delta N = \exp(-(z - z_0)^2/\sigma^2)$, $\sigma \sim 200\ \mathrm{km}$, $\varepsilon \sim 10^{-5}$, integral is $\sim 4\ \mathrm{m}$
  - ε^2 term is $\sim 10^{-5}\ \mathrm{m}$