

Analytical distribution functions for driven PIC simulations (And relativistic double-adiabatic equations)

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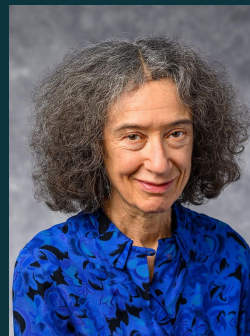
Acknowledgements

2



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A Tale of Two Outcomes...

Two main results of FL, Tran & Zweibel 2026:

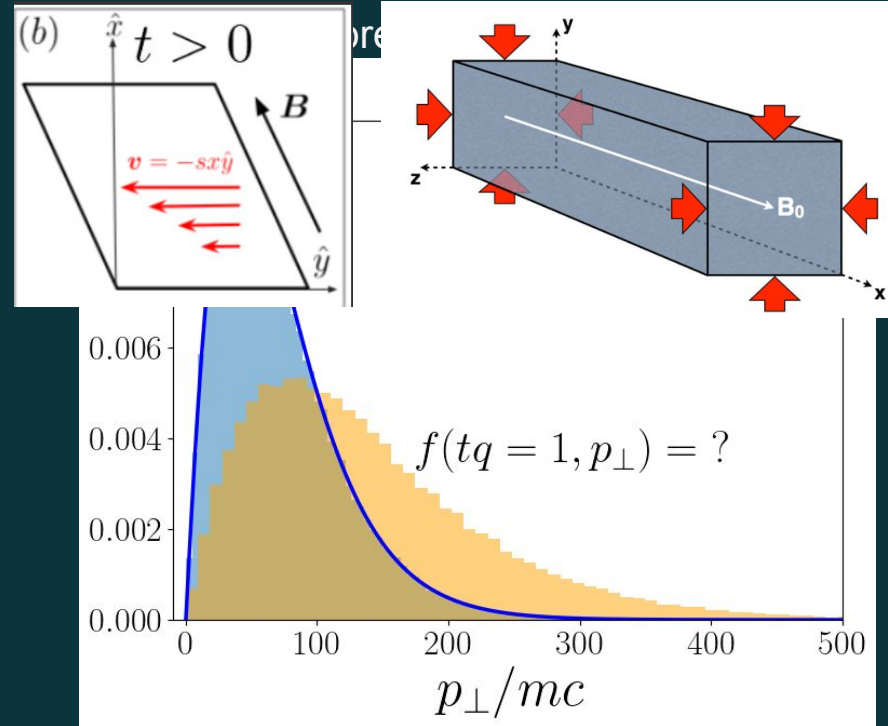
- **Time-dependent anisotropic, gyrotropic distribution functions**
 - General form of distribution subject to time-dependent adiabatic motions.
 - Newly proposed anisotropic, relativistic Maxwell-Jüttner (e.g. *Schlickeiser 2004, Treumann 2015, Livadiotis 2016*)
- **Extending double-adiabatic equations of state to relativistic regime**
 - Analytical, closed form for Ultrarelativistic case.
 - Numerically obtained for the generic relativistic case.

Does all this agree with PIC simulations??

The Initial Spark: Evolution of distributions in PIC

In a driven PIC simulation of a uniform, magnetized plasma (e.g. shearing/compressing/pulsating box):

- How does the distribution function evolve during the adiabatic phase of the motion?
- Can we obtain an analytical expression for it? (e.g. anisotropic Maxwell-Jüttner)



Relativistic Astrophysical Plasmas

- Weakly collisional/collisionless, magnetized, and turbulent.
- Several species can be relativistic (e.g. cosmic rays, accelerated populations, relativistic temperatures)
- Departures from thermodynamic equilibrium are common.
- Pressure anisotropy: $P_{\perp} \neq P_{\parallel}$

Event Horizon Telescope



Chandra X-ray Observatory



Chandra X-ray Observatory
Hubble Space Telescope
Spitzer Space Telescope



Pressure Anisotropy and the CGL Equations

- Slow dynamics in magnetized, collisionless plasma: magnetic moment and longitudinal action are **adiabatic invariants**.
- In absence of heat fluxes and FLR effects: **CGL equations** (*Chew, Goldberger, Low 1956*).
- CGL can be obtained from moments of a well known distribution: **Bi-Maxwellian**.
- Important source of free energy for kinetic instabilities (mirror, firehose, ion cyclotron, whistler, etc.)

$$\dot{B}/B \ll \Omega_\alpha$$

$$\dot{n}_\alpha/n_\alpha \ll \Omega_\alpha$$

$$\mathcal{M} = \frac{v_{\perp,\alpha}^2}{2m_\alpha B} \sim \text{const.}$$

$$\mathcal{J} = \oint v_{\parallel,\alpha} d\ell \sim \text{const.}$$

$$\frac{d}{dt} \left(\frac{P_{\perp,\alpha}}{n_\alpha B} \right) = 0$$

$$\frac{d}{dt} \left(\frac{P_{\parallel,\alpha} B^2}{n_\alpha^3} \right) = 0$$

Pressure Anisotropy and the CGL Equations

- Slow dynamics in magnetized, collisionless plasma: magnetic moment and longitudinal action are **adiabatic invariants**.
- In absence of heat flux effects: **Low**
- CGL well known
- Important instabilities: magnetic, ion cyclotron, whistler,

All working well in the non-relativistic regime.
But what happens in the relativistic case?

$$\dot{B}/B$$

$$\frac{d}{dt} \left(\frac{P_{\perp, \alpha}}{n_{\alpha} B} \right) = 0$$

$$\frac{d}{dt} \left(\frac{P_{\parallel, \alpha} B^2}{n_{\alpha}^3} \right) = 0$$

Let's find $\mathbf{f}$: The Relativistic Drift Kinetic Equation

- Periodic, uniform, magnetised, collisionless flux tube of plasma of radius r and length l , with no scattering mechanism active and zero heat fluxes.
- Evolution of a gyrotropic distribution $f(t, p_{\perp}, p_{\parallel})$ is given by (e.g. Kulsrud 1983, 2005)

$$\frac{\partial f}{\partial t} + \frac{dp_{\perp}}{dt} \frac{\partial f}{\partial p_{\perp}} + \frac{dp_{\parallel}}{dt} \frac{\partial f}{\partial p_{\parallel}} = 0$$

- We assume very long spatial scales $L_n s_n, L_B s_B \gg v_{th}$, so spatial derivatives of $\mathbf{f}$ can be neglected.

$$\begin{aligned} s_n &\equiv d \ln n / dt & L_n &\equiv n / \nabla n \\ s_B &\equiv d \ln B / dt & L_B &\equiv B / \nabla B \end{aligned}$$

- A more general approach via a Lagrangian formalism is possible, and invoking parity in $\mathbf{f}$ in order to neglect heat fluxes. (*Wierzchucka et al. 2026*)

Let's find f : The Relativistic Drift Kinetic Equation

$$p = \sqrt{p_{\perp}^2 + p_{\parallel}^2}$$

$$\mu = p_{\parallel}/p$$

- Assume well magnetised plasma: $\Omega/s_n \gg 1, \Omega/s_B \gg 1$ $s_n \equiv d \ln n / dt$
 $s_B \equiv d \ln B / dt$
- Magnetic moment and longitudinal action are adiabatic invariants:

$$\mathcal{M} = \frac{p_{\perp, \alpha}^2}{2mB} \sim \text{const.} \quad \mathcal{J} = \oint p_{\parallel, \alpha} d\ell \sim \text{const.}^*$$

- Conservation of adiabatic invariants, particle number, and magnetic flux allow to write, using momentum spherical coordinates (e.g. *Lichko et al. 2017*)

$$\frac{\partial f}{\partial t} + \left[\frac{1}{2}(3\mu^2 - 1) \underbrace{\left(\frac{2}{3} \frac{\dot{n}}{n} - \frac{\dot{B}}{B} \right)}_{-\hat{b}\hat{b} : \nabla \mathbf{u} + \frac{1}{3} \nabla \cdot \mathbf{u}} + \frac{\dot{n}}{3n} \right] p \frac{\partial f}{\partial p} + \left[\frac{3}{2} \left(\frac{2}{3} \frac{\dot{n}}{n} - \frac{\dot{B}}{B} \right) \mu(1 - \mu^2) \right] \frac{\partial f}{\partial \mu} = 0$$

$$\underbrace{\left(\frac{2}{3} \frac{\dot{n}}{n} - \frac{\dot{B}}{B} \right)}_{-\frac{1}{3} \nabla \cdot \mathbf{u}}$$

* : See *Wierzchucka et al. 2026* for a nice generalization of the use of $\mathcal{J}$.

Let's find f : The Relativistic Drift Kinetic Equation

$$\frac{\partial f}{\partial t} + \left[\frac{1}{2}(3\mu^2 - 1) \left(\frac{2}{3} \frac{\dot{n}}{n} - \frac{\dot{B}}{B} \right) + \frac{\dot{n}}{3n} \right] p \frac{\partial f}{\partial p} + \left[\frac{3}{2} \left(\frac{2}{3} \frac{\dot{n}}{n} - \frac{\dot{B}}{B} \right) \mu(1 - \mu^2) \right] \frac{\partial f}{\partial \mu} = 0$$

- Can be solved by the method of characteristics yielding (e.g. *Zhdankin, Kunz & Uzdensky 2023*)

$$f(t, p, \mu) = F \left(p \left(\frac{n}{n_0} \right)^{-1/3} \left\{ \left(\frac{R}{R_0} \right)^{-1} + \mu^2 \left[\left(\frac{R}{R_0} \right)^2 - \left(\frac{R}{R_0} \right)^{-1} \right] \right\}^{1/2}, \left[1 + \frac{1 - \mu^2}{\mu^2} \left(\frac{R}{R_0} \right)^{-3} \right]^{-1/2} \right)$$

For any gyrotropic initial distribution F .

$$R(t) \equiv n^{-2/3} B$$

We consider three initial distributions:

1. Non-relativistic Maxwellian
2. Maxwell-Jüttner
3. Ultrarelativistic Maxwell-Jüttner

Benchmark: Initial Non-relativistic Maxwellian

Initial F:
$$F(v) = \frac{n_0}{\pi^{3/2} v_{th}^3} \exp\left(-\frac{v^2}{v_{th}^2}\right)$$

Leads to:
$$f(t, v, \mu) = \frac{n_0}{\pi^{3/2} v_{th}^3} \exp\left(-\frac{v_{\perp}^2}{v_{th}^2 (B/B_0)} - \frac{v_{\parallel}^2}{v_{th}^2 (n/n_0)^2 (B/B_0)^{-2}}\right)$$

$$f(t, v, \mu) = \frac{n_0}{\pi^{3/2} v_{th}^3} \exp\left(-\frac{mv_{\perp}^2}{k_B T_{\perp}(t)} - \frac{mv_{\parallel}^2}{k_B T_{\parallel}(t)}\right) \quad \textbf{Bi-Maxwellian form}$$

We recover CGL:
$$P_{\perp} = m \int \frac{v^2(1 - \mu^2)}{2} f(t, v, \mu) d^3v = P_0 \left(\frac{n}{n_0}\right) \left(\frac{B}{B_0}\right)$$

$$P_{\parallel} = m \int v^2 \mu^2 f(t, v, \mu) d^3v = P_0 \left(\frac{n}{n_0}\right)^3 \left(\frac{B}{B_0}\right)^{-2},$$

Relativistic Case: Maxwell-Jüttner

Initial F:
$$F(p) = \frac{n_0}{4\pi m^3 c^3 K_2(1/\theta)} \exp \left(\frac{1}{\theta} \sqrt{1 + \frac{p_\perp^2}{m^2 c^2} + \frac{p_\parallel^2}{m^2 c^2}} \right) \quad \theta \equiv k_B T_0 / mc^2$$

Leads to a new time-dependent, **anisotropic Maxwell-Jüttner** (e.g. Schlickeiser 2004, Treumann 2015, Livadiotis 2016):

$$f(t, p_\perp, p_\parallel) = \frac{n_0}{4\pi m^3 c^3 \theta K_2(1/\theta)} \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_\perp^2 / m^2 c^2}{(B/B_0)} + \frac{p_\parallel^2 / m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

Can this expression describe the **adiabatic evolution** of a Maxwell-Jüttner in driven PIC simulations?

Numerical Validation: Driven PIC Simulations

- Continuously driven, 2.5D fully kinetic PIC simulations of shearing and compressing boxes using TRISTAN-MP (*Riquelme et al. 2012, Sironi & Narayan 2015*).
- Ions and electrons initialized with Maxwell-Jüttner distributions and equal temperatures $T_i = T_e$.
- We avoid excitation of any kinetic instability. Only **adiabatic evolution** is captured.
- Physical parameters:

$k_B T_i^{\text{init}} / m_i c^2$
0.2
30

β_i^{init}
0.05
0.5
5

m_i / m_e
8
64
1836

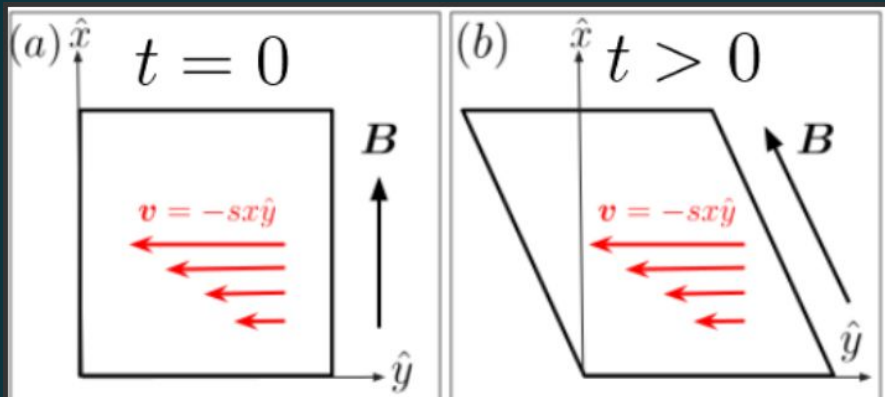
$\Omega_i / s, \Omega_i / q$
800
3200

Numerical Validation: Driven PIC Simulations

Shearing box: (Riquelme et al. 2012)

- Velocity shear imposed: $\mathbf{v} = -sx\hat{y}$
- Flux freezing implies $B(t) = B_0\sqrt{1 + s^2t^2}$

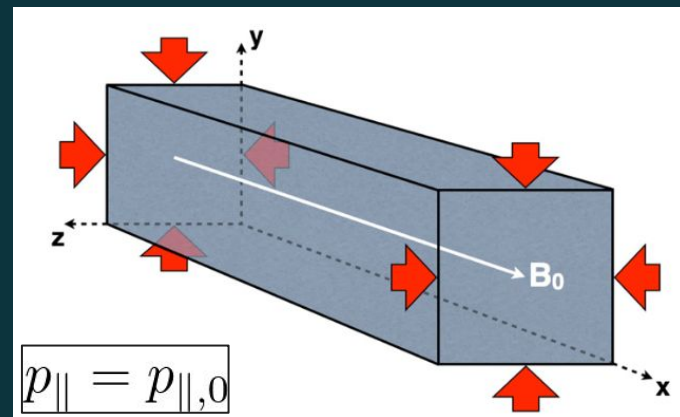
$$\dot{B}/B \neq 0, \dot{n}/n = 0$$



Compressing box: (Sironi & Narayan 2015)

- Compression motion imposed in perpendicular directions of B .
- Flux freezing implies $n(t) = n_0(1 + qt)^2$
 $B(t) = B_0(1 + qt)^2$

$$\dot{B}/B \neq 0, \dot{n}/n \neq 0$$



Results: Adiabatic time evolution of the distribution

Recall our time-dependent, anisotropic
Maxwell-Jüttner:

$$f(t, p_{\perp}, p_{\parallel}) = \frac{n_0}{4\pi m^3 c^3 \theta K_2(1/\theta)} \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)} + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

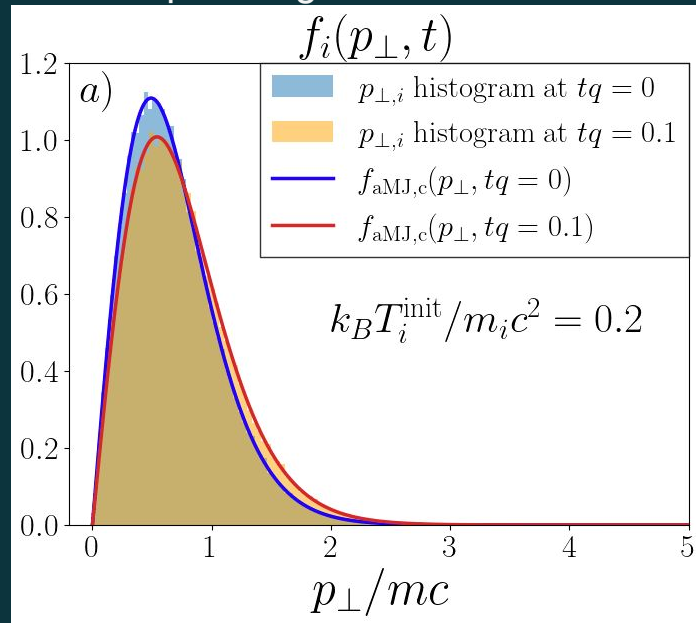
Marginalizing in each momentum coordinate:

$$f_{\text{aMJ}}(t, p_{\perp}) = \frac{n_0}{m^2 c^2 \theta K_2(1/\theta)} \frac{(n/n_0)}{(B/B_0)} p_{\perp} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)}} K_1 \left(\frac{1}{\theta} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)}} \right)$$

$$f_{\text{aMJ}}(t, p_{\parallel}) = \frac{n_0}{2mc K_2(1/\theta)} \left(\sqrt{1 + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} + \theta \right) \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

Directly compare with simulations!

Compressing box PIC simulation



Results: Adiabatic time evolution of the distribution

Recall our time-dependent, anisotropic Maxwell-Jüttner:

$$f(t, p_{\perp}, p_{\parallel}) = \frac{n_0}{4\pi m^3 c^3 \theta K_2(1/\theta)} \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)} + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

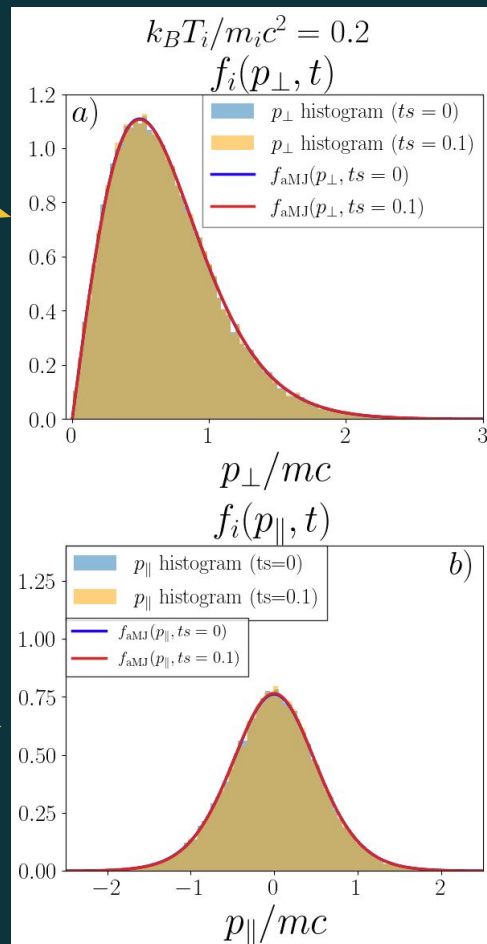
Marginalizing in each momentum coordinate:

$$f_{\text{aMJ}}(t, p_{\perp}) = \frac{n_0}{m^2 c^2 \theta K_2(1/\theta)} \frac{(n/n_0)}{(B/B_0)} p_{\perp} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)}} K_1 \left(\frac{1}{\theta} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)}} \right)$$

$$f_{\text{aMJ}}(t, p_{\parallel}) = \frac{n_0}{2mc K_2(1/\theta)} \left(\sqrt{1 + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} + \theta \right) \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

Directly compare with simulations!

Shearing box PIC simulation



Pressure Anisotropy from Anisotropic Maxwell-Jüttner?

Recall our newly proposed Anisotropic Maxwell-Jüttner:

$$f(t, p_{\perp}, p_{\parallel}) = \frac{n_0}{4\pi m^3 c^3 \theta K_2(1/\theta)} \exp \left(-\frac{1}{\theta} \sqrt{1 + \frac{p_{\perp}^2/m^2 c^2}{(B/B_0)} + \frac{p_{\parallel}^2/m^2 c^2}{(n/n_0)^2 (B/B_0)^{-2}}} \right)$$

Relativistic moments: (spherical coordinates)

$$P_{\perp} = \frac{n_0 c}{2m^4 c^4 \theta K_2(1/\theta)} \int \frac{1}{2} \frac{p^4 (1 - \mu^2)}{\sqrt{1 + p^2/m^2 c^2}} \exp \left(-\frac{1}{\theta} \sqrt{1 + A_{\mu} \frac{p^2}{m^2 c^2}} \right) dp d\mu$$

$$P_{\parallel} = \frac{n_0 c}{2m^4 c^4 \theta K_2(1/\theta)} \int \frac{p^4 \mu^2}{\sqrt{1 + p^2/m^2 c^2}} \exp \left(-\frac{1}{\theta} \sqrt{1 + A_{\mu} \frac{p^2}{m^2 c^2}} \right) dp d\mu$$

$$A_{\mu} \equiv (n/n_0)^{-2/3} (R/R_0)^{-1} \{1 + \mu^2 [(R/R_0)^3 - 1]\}$$

$$R(t) \equiv n^{-2/3} B$$

Ultrarelativistic case

Initial F:
$$f_0(p) = \frac{n_0}{8\pi p_T^3} \exp\left(-\frac{p}{p_T}\right) \quad p_T \equiv k_B T/c$$

Leads to:
$$f(t, p, \mu) = \frac{n_0}{8\pi p_T^3} \exp\left(-\frac{p}{p_T} \left(\frac{n}{n_0}\right)^{-1/3} \left(\frac{R}{R_0}\right)^{-1/2} \left\{1 + \mu^2 \left[\left(\frac{R}{R_0}\right)^3 - 1\right]\right\}^{1/2}\right)$$

Taking relativistic moments: **Ultrarelativistic Double-adiabatic equations:**

$$P_{\perp} = \frac{3}{4} P_0 \left(\frac{n}{n_0}\right)^{4/3} \left(\frac{R}{R_0}\right)^2 \left\{ \frac{(R/R_0)^3 - 2}{[(R/R_0)^3 - 1]^{3/2}} \arctan\left(\sqrt{(R/R_0)^3 - 1}\right) + \frac{1}{(R/R_0)^3 - 1} \right\}$$

$$P_{\parallel} = \frac{3}{2} P_0 \left(\frac{n}{n_0}\right)^{4/3} \left(\frac{R}{R_0}\right)^2 \left\{ \frac{\arctan\left(\sqrt{(R/R_0)^3 - 1}\right)}{[(R/R_0)^3 - 1]^{3/2}} - \frac{1}{(R/R_0)^3 [(R/R_0)^3 - 1]} \right\},$$

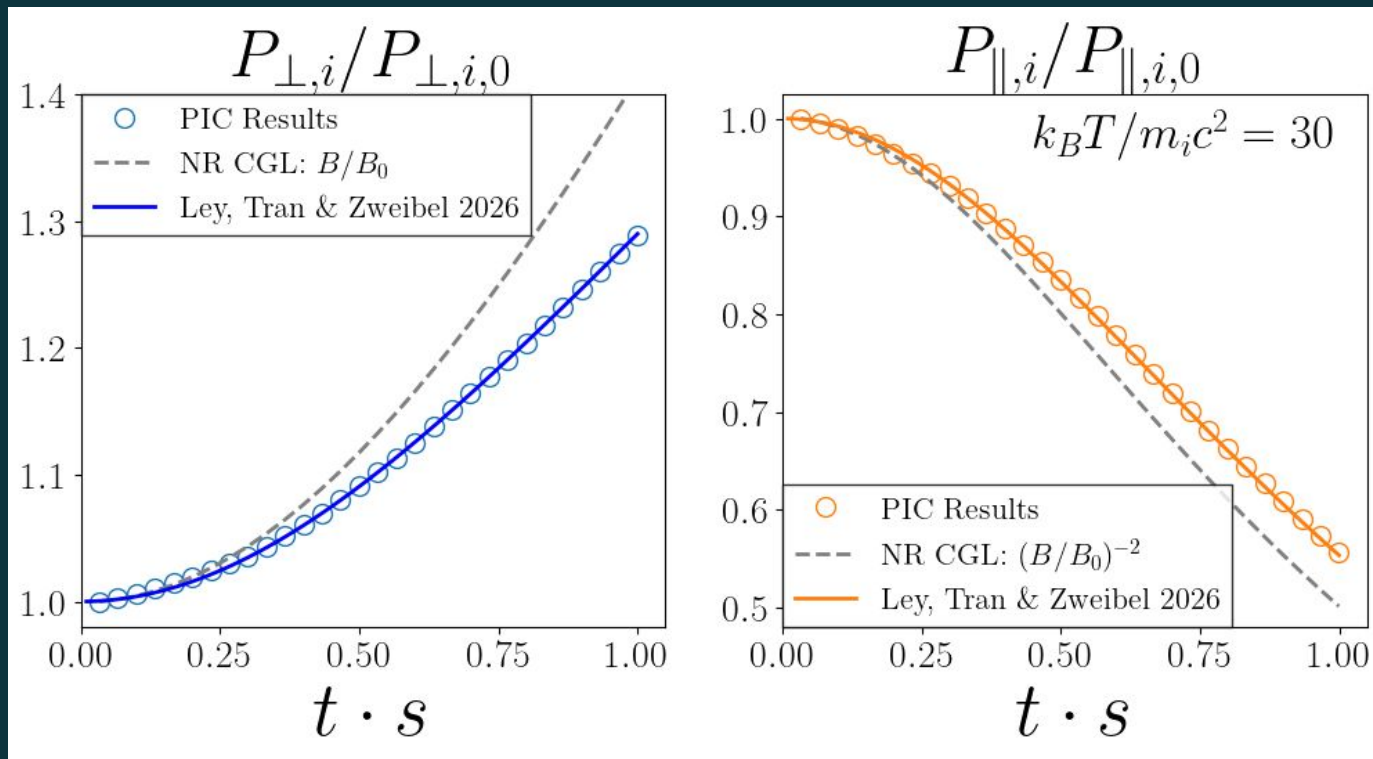
For $R > 1$

See also *Wierzchucka et al. 2026*

PIC Results: Shearing box

1. Ultrarelativistic case $k_B T_i^{\text{init}}/m_i c^2 = 30 |m_i/m_e = 1836| \beta_i^{\text{init}} = 0.5$

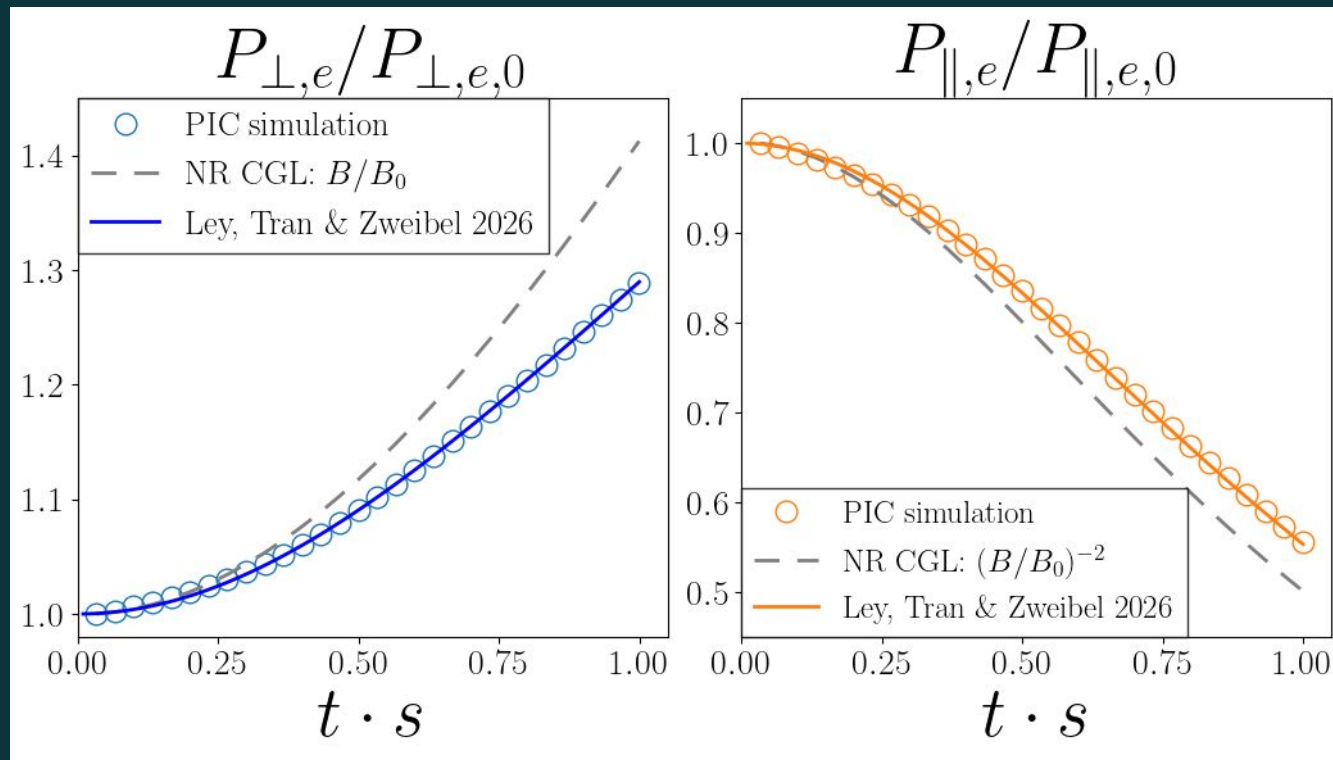
Ion
case:



PIC Results: Shearing box

1. Ultrarelativistic case $k_B T_e^{\text{init}}/m_e c^2 = 55,080$ $|m_i/m_e = 1836| \beta_e^{\text{init}} = 0.5$

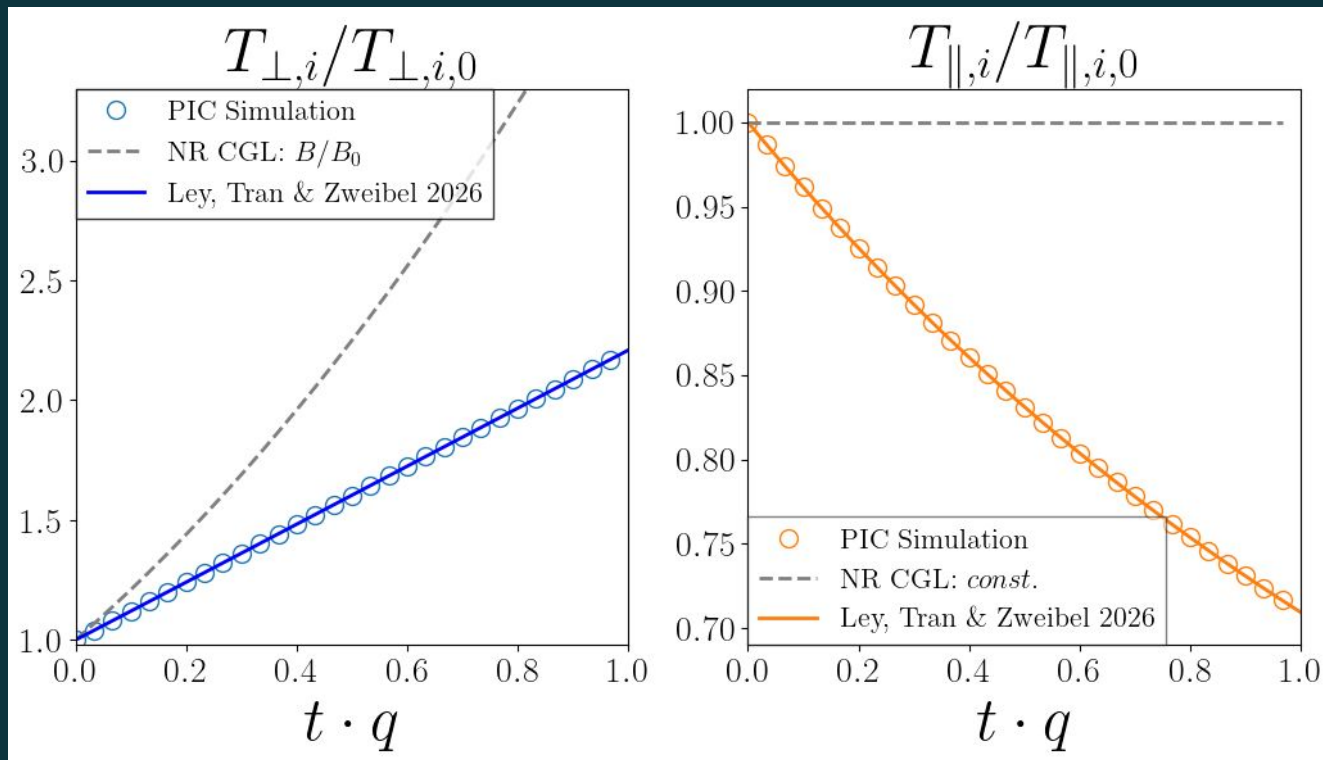
Electron
case:



PIC Results: Compressing box

1. Ultrarelativistic case $k_B T_i^{\text{init}}/m_i c^2 = 30 |m_i/m_e = 1836| \beta_i^{\text{init}} = 0.5$

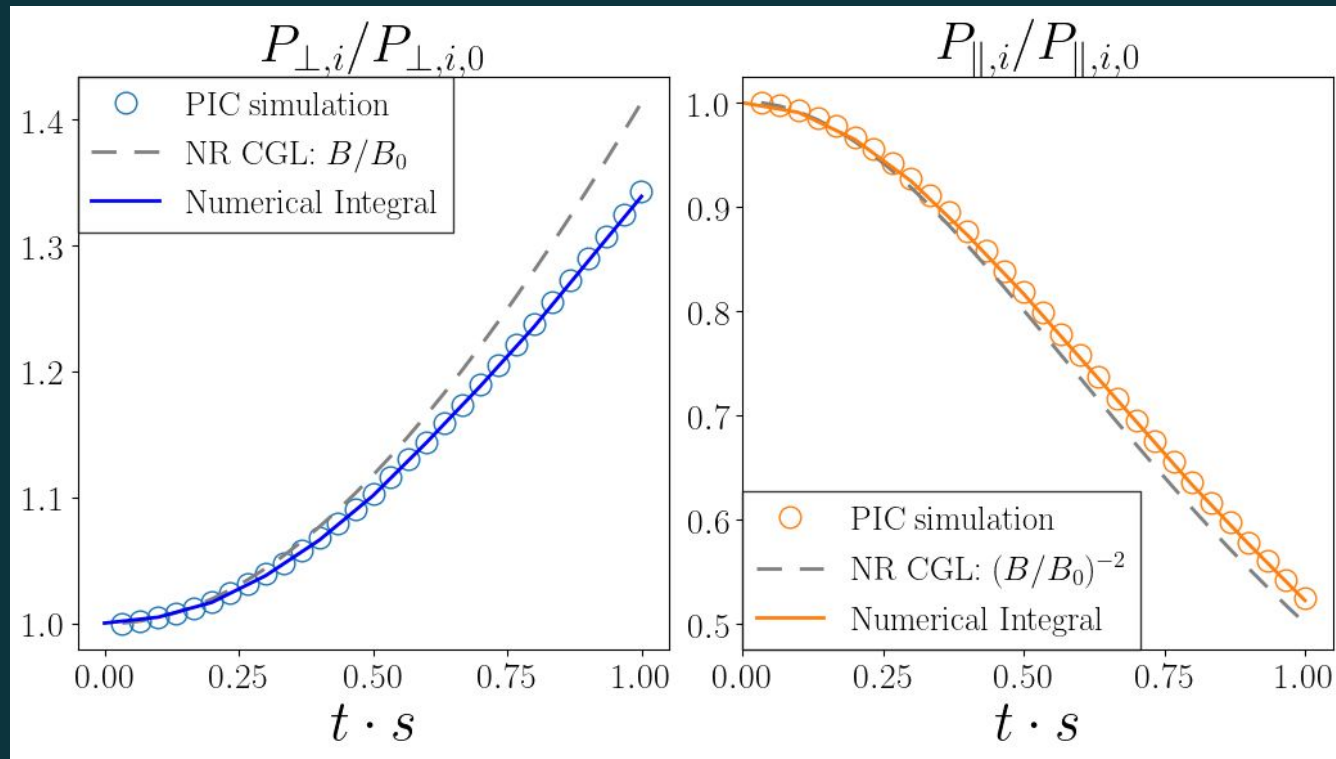
Ion
case:



PIC Results: Shearing box

2. Relativistic case: $k_B T_i^{\text{init}} / m_i c^2 = 0.2 \mid m_i / m_e = 8 \mid \beta_i^{\text{init}} = 0.5$

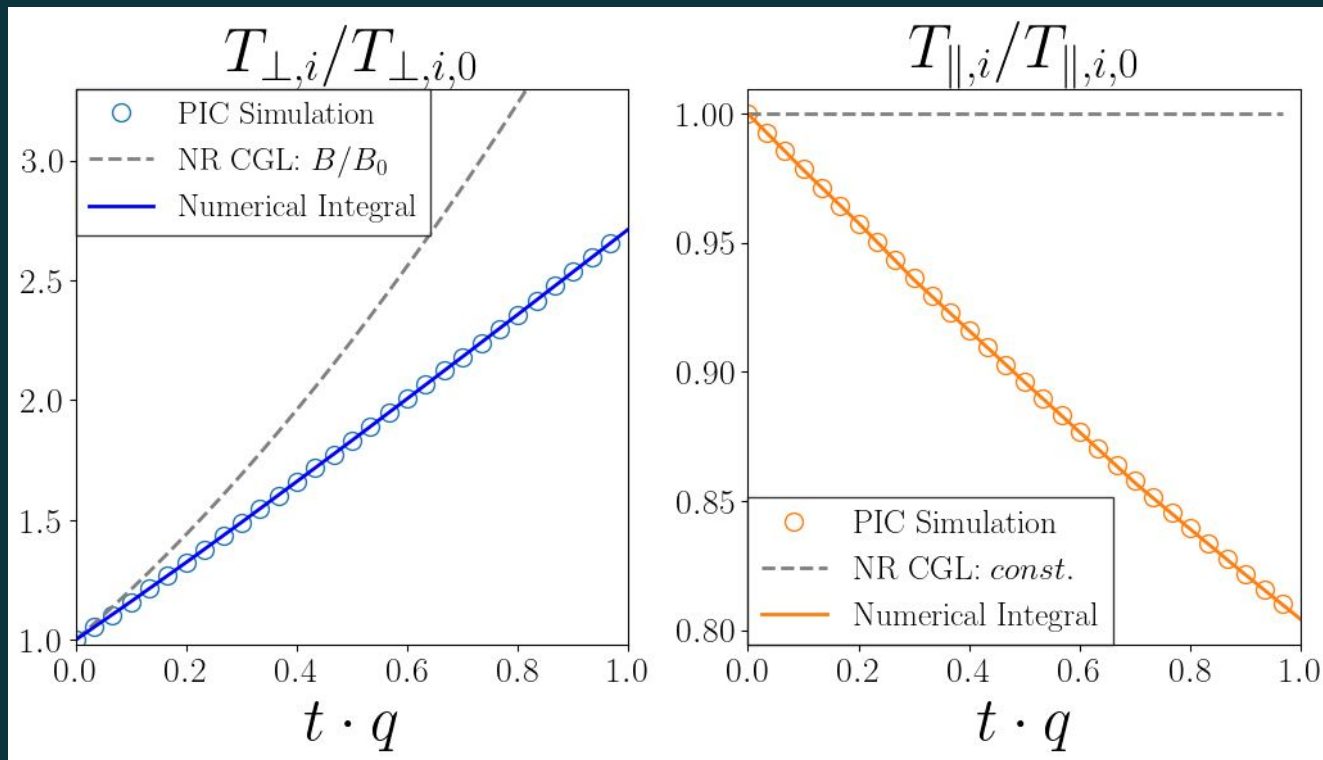
Ion
case:



PIC Results: Compressing box

2. Relativistic case: $k_B T_i^{\text{init}} / m_i c^2 = 0.2 \mid m_i / m_e = 8 \mid \beta_i^{\text{init}} = 0.5$

Ion
case:



Applications

- Extended MHD theories for relativistic collisionless plasmas (with non-relativistic bulk flows):

- Anisotropic relativistic distributions.
 - Closure relations.

$$\frac{d}{dt} \left(\frac{P_{\perp}}{B^2 I_{\perp}(A)} \right) = 0, \quad \frac{d}{dt} \left(\frac{P_{\parallel} B}{n^2 I_{\parallel}(A)} \right) = 0.$$

Wierchucka et al. 2026

- Extend results from *Chandra et al. 2015* for the evolution of the pressure anisotropy and Braginskii limit in the ultrarelativistic regime:

$$\frac{1}{P} \frac{d\Delta P}{dt} = \mathcal{A} \frac{\dot{B}}{B} + \mathcal{C} \frac{\dot{n}}{n}$$

$$\begin{aligned} \mathcal{A} &= \frac{3}{2} \Psi(r) \left[\frac{(r^3 + 2)(r^3 + 8)}{2(r^3 - 1)^{5/2}} \arctan(\sqrt{r^3 - 1}) + \frac{r^6 - 32r^3 + 4}{2r^3(r^3 - 1)^2} \right] \\ \mathcal{C} &= -3\Psi(r) \left[\frac{r^3(10 - r^3)}{2(r^3 - 1)^{5/2}} \arctan(\sqrt{r^3 - 1}) - \frac{r^6 + 12r^3 - 4}{2r^3(r^3 - 1)^2} \right], \\ \Psi(r) &= \frac{r^3 \sqrt{r^3 - 1}}{r^3 \arctan(\sqrt{r^3 - 1}) + \sqrt{r^3 - 1}}, \end{aligned}$$

In the small anisotropy limit: $r \rightarrow 1$

$$\frac{1}{P} \frac{d\Delta P}{dt} = \frac{12}{5} \frac{\dot{B}}{B} - \frac{8}{5} \frac{\dot{n}}{n} = 4 \frac{d}{dt} \ln \left(\frac{B^3}{n^2} \right) + \mathcal{O}([\Delta P/P]^2)$$

(eq. 108 in Chandra et al. 2015)

Applications

- Modified relativistic fast magnetosonic speed in *Ugarov, Zhdankin & Arrò 2026*.
- Linearized relativistic ideal MHD system + UR double-adiabatic pressures:

$$\begin{aligned}
 \rho_1 &= \frac{k}{\omega} \rho_0 v_{1x} \\
 w_1 c - \delta P_{\perp} + \frac{B_0 B_1}{4\pi} &= \frac{ck}{\omega} \left(w_0 c^2 v_{1x} + \frac{E_{1y} B_0}{4\pi} \right) \\
 w_0 \mathbf{v}_1 + \frac{\mathbf{E}_1 \times B_0 \hat{z}}{4\pi c} &= \hat{x} \frac{k}{\omega} \left(\delta P_{\perp} + \frac{B_0 B_1}{4\pi} \right) \\
 B_{1z} &= \frac{ck}{\omega} E_{1y} \\
 \mathbf{E}_1 &= -\frac{\mathbf{v}_1 \times \mathbf{B}_0}{c} \\
 \delta P_{\perp} &= \frac{4}{5} P_0 \left(\frac{\rho_1}{\rho_0} + \frac{B_1}{B_0} \right) \\
 \delta P_{\parallel} &= \frac{4}{5} P_0 \left(3 \frac{\rho_1}{\rho_0} - 2 \frac{B_1}{B_0} \right)
 \end{aligned}$$

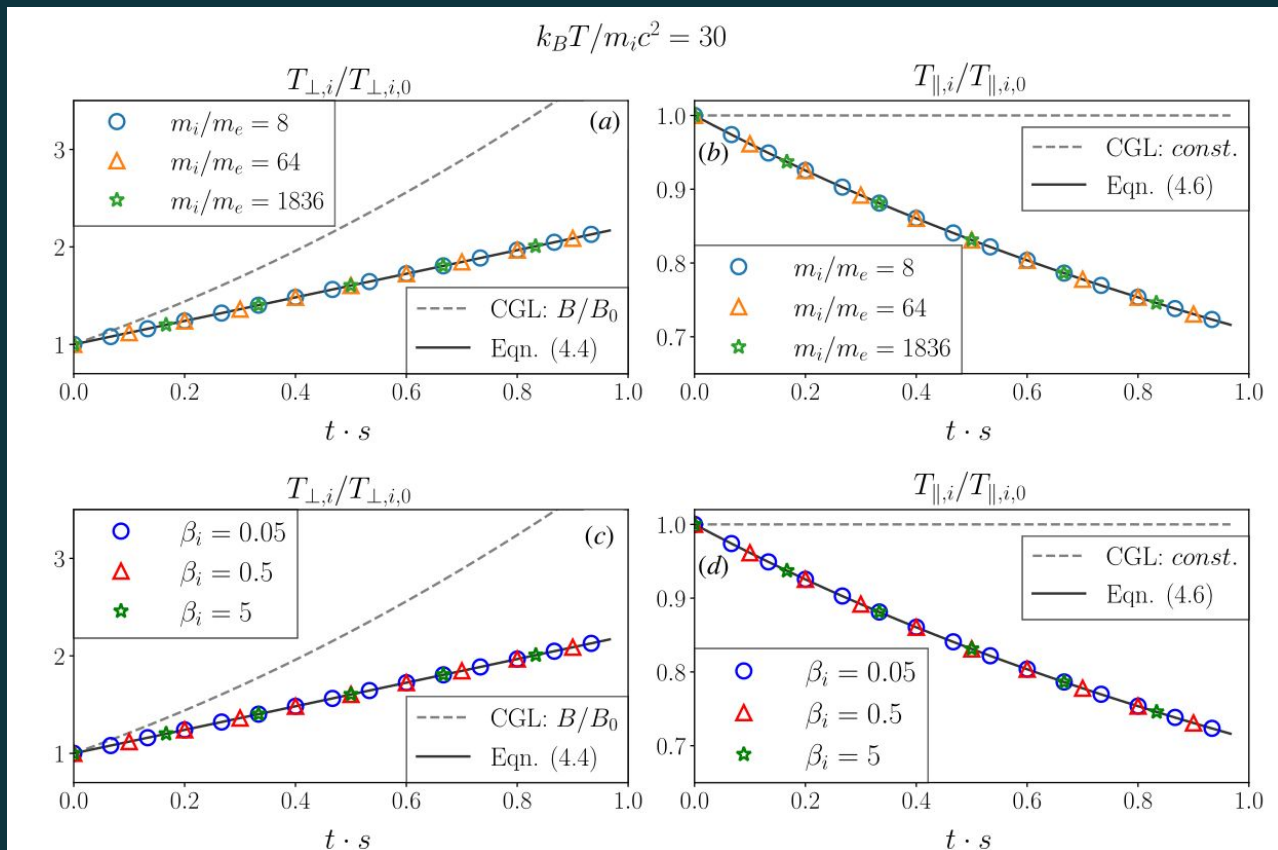
$$\frac{(R/R_0)^3 - 2}{(R/R_0)^3 - 1} \ar \frac{\omega^2}{k^2} = \frac{(2/5)c^2 + u_A^2}{1 + u_A^2/c^2} \cdot \left\{ \frac{\tan \left(\sqrt{(R/R_0)^3 - 1} \right)}{[(R/R_0)^3 - 1]^{3/2}} - \frac{1}{(R/R_0)^3 [(R/R_0)^3 - 1]} \right\}$$

leading order term of eq. 11 in *Ugarov et al. 2026*

Results are Independent of plasma β and mass ratio

26

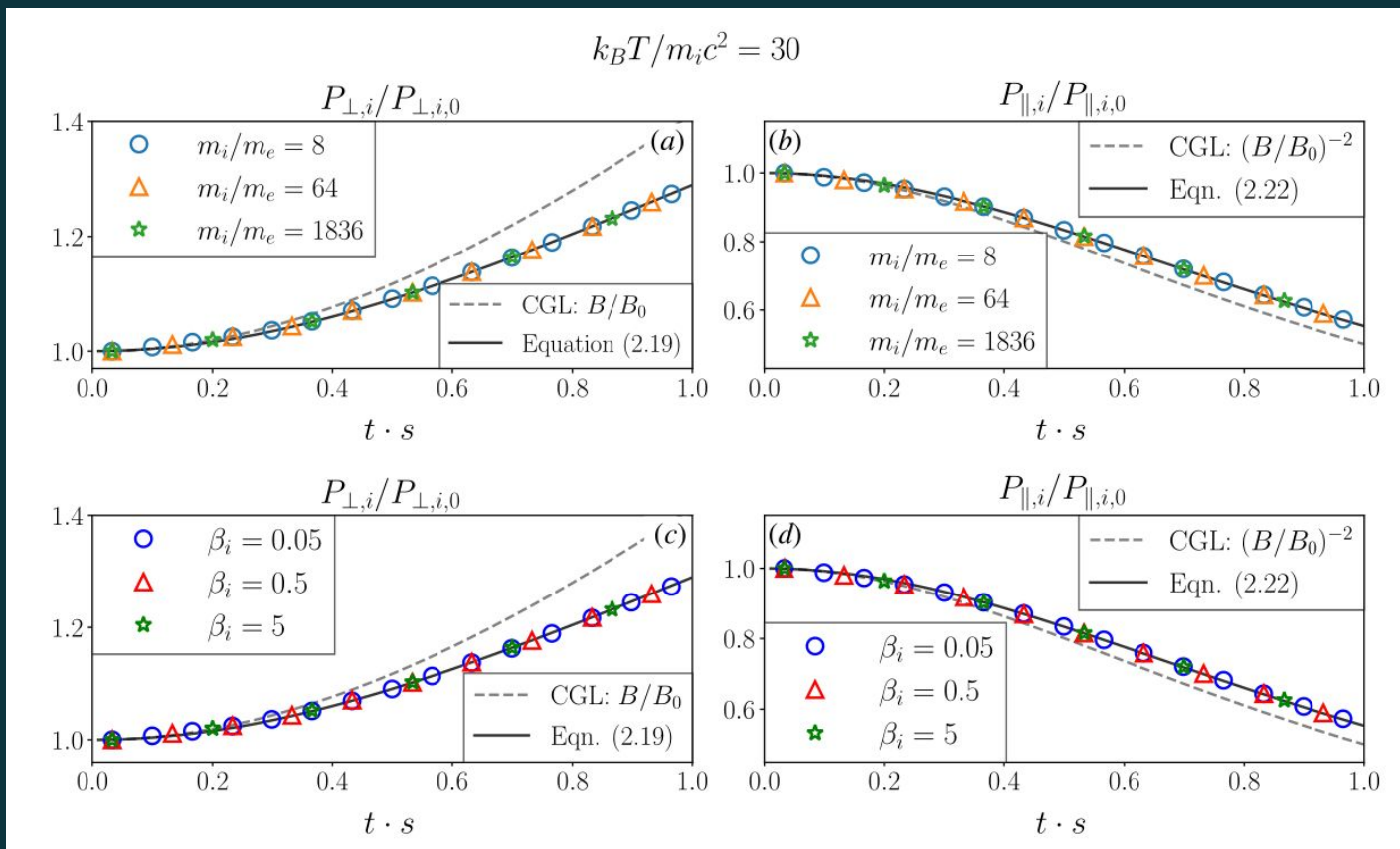
Compressing Box:



Results are Independent of plasma β and mass ratio

27

Shearing Box:



Applications: Event-horizon scale accretion flows

Dhruv, Sironi, Davelaar, and Tran 2026: Kinetic Instabilities in Magnetically Arrested Disks

$$k_B T_i^{\text{init}} / m_i c^2 = 0.05$$

$$k_B T_e^{\text{init}} / m_e c^2 = 91.8$$

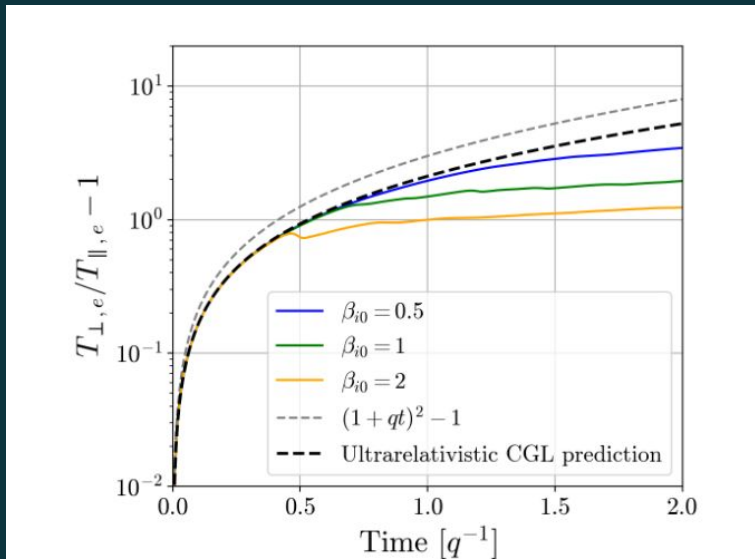


Figure C1. Time series of the electron temperature anisotropy $A_{T,e} \equiv T_{\perp,e}/T_{\parallel,e} - 1$ for a representative sample of simulations with $T_{e0}/T_{i0} = 1$ and $\Theta_{i0} = 0.05$, at varying β_{i0} (colored solid lines; blue corresponds to the fiducial simulation). The gray dashed line shows the nonrelativistic double-adiabatic prediction, $(1 + qt)^2 - 1$, while the black dashed line shows the ultrarelativistic prediction for a compressing box (F. Ley et al. 2026; A. Wierzhucka et al. 2026). The simulated anisotropy follows the ultrarelativistic prediction until instabilities set in and drive the anisotropy toward the marginal stability threshold.

Conclusions

- By solving homogeneous drift kinetic equation a general, time-dependent anisotropic distribution function is obtained.
 - New **anisotropic Maxwell-Jüttner** proposed.
- We have extended the **double-adiabatic** equations of state to **ultrarelativistic** (analytical) and **relativistic regimes** (numerical).
- Evolution of the simulation VDF agrees with newly proposed anisotropic Maxwell-Jüttner distribution.
- We validate the relativistic double-adiabatic equations with continuously driven PIC simulations of shearing and compressing boxes.
- Applications and future work:
 - Closures for extended MHD (with NR bulk flows)
 - Relativistic instability thresholds for mirror, firehose, ion-cyclotron, whistlers, etc.
 - Different initial distributions, e.g., power-law, kappa.

Bonus

Previous works

- *Gedalin 1991 and Gedalin & Oiberman 1995*: Most general state equations for a relativistic, anisotropic, collisionless plasma:

$$P_{\parallel} = n^2 \frac{\partial}{\partial n} \left(\frac{U}{n} \right)$$

$$P_{\perp} = P_{\parallel} + nB \frac{\partial}{\partial B} \left(\frac{U}{n} \right)$$

where $U = c \int p^0 f(p_{\perp}^2, p_{\parallel}^2) d^3 p = \int \gamma m c^2 f(p_{\perp}^2, p_{\parallel}^2) d^3 p$ is the internal energy density,

and $f(p_{\perp}^2, p_{\parallel}^2)$ gyrotropic distribution satisfying the relativistic Vlasov equation.

- Any future attempt must obey these relations. Can we obtain one explicit form of $f(p_{\perp}^2, p_{\parallel}^2)$?

$$R(t) \equiv n^{-2/3} B$$

Ultrarelativistic case

If we calculate the internal energy density U :

$$U = \frac{3}{2} P_0 \left(\frac{n}{n_0} \right)^{4/3} \left(\frac{R}{R_0} \right)^2 \left\{ \frac{\arctan \left(\sqrt{(R/R_0)^3 - 1} \right)}{\sqrt{(R/R_0)^3 - 1}} + \frac{1}{(R/R_0)^3} \right\} \quad \text{For } R > 1$$

- **Ultrarelativistic Double-adiabatic equations** satisfy most general state equations from *Gedalin 1991, Gedalin & Oiberman 1995*.

$$P_{\parallel} = n^2 \frac{\partial}{\partial n} \left(\frac{U}{n} \right)$$

$$P_{\perp} = P_{\parallel} + nB \frac{\partial}{\partial B} \left(\frac{U}{n} \right)$$

- *Wierzychucka et al. 2026* also show agreement with asymptotic limits of *Gedalin & Oibermann 1995*: $P_{\perp} \gg P_{\parallel}, P_{\perp} \sim P_{\parallel}, P_{\perp} \ll P_{\parallel}$